OCTOBER, 1890.

ANNALS OF MATHEMATICS.

ORMOND STONE, Editor.
W. M. THORNTON, } Associate Editors.
R. S. WOODWARD, }

OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 5, Number 5.

ALL COMMUNICATIONS should be addressed to ORMOND STONE, University Station, Charlottesville, Va., U. S. A.
Entered at the Post Office as second-class mail matter.

ANNALS OF MATHEMATICS.

VOL. V. OCTOBER, 1890. NO. 5.

THE SECULAR PERTURBATIONS OF TWO PLANETS MOVING IN THE SAME PLANE; WITH APPLICATION TO JUPITER AND SATURN.

By DR. G. W. HILL, Washington, D. C.

The solution of this problem, when we restrict ourselves to the first powers of the eccentricities, is as old as Lagrange, and is well known. Leverrier, in going over this ground, attempted to include the effect of the terms of three dimensions with respect to eccentricities and inclinations.* But when his method was applied to the four interior planets of the solar system it led to results that were nugatory. This method being that of successive approximations, the expressions for the unknowns obtained in the simplest form of the investigation were substituted in the terms of three dimensions; in consequence, he arrived at the same linear differential equations as before, but now augmented by known terms. His difficulty, in the case of the four interior planets, arose from the appearance in the results of integrating divisors which might receive very small, or even zero, values within the range of uncertainty of the values of the planetary masses.

As far as the general question is concerned, no one has attempted to push the investigation further. Under these circumstances I have thought it might be well to treat as completely as we can the very simple case where we have only two planets executing their motions in the same plane. Although we see here at a glance that the problem is reducible to quadratures, yet this taken by itself does not constitute a practical solution. Some difficulties are encountered in deriving from the quadratures series suitable for calculating the values of the unknowns. These difficulties I have succeeded in surmounting by a process which would not suggest itself, I think, at first sight.

In the application which I have made to the case of Jupiter and Saturn with neglected mutual inclination, I have carried the approximation to quan-

*Annales de l'Observatoire de Paris, Tom. II, pp. 105–170 and pp. [38]–[51].

tities of the fifth order, inclusive; and it is not difficult to see what must be done if it is desired to go further.

I.

The first thing to be done in this investigation is to find a proper development of the potential or perturbative function. Quantities belonging to the interior planet will be denoted by symbols without an accent, and those belonging to the exterior by symbols having an accent. Let, then, m, r, a, g, u, and f denote severally the mass of the planet, the radius, the semi-axis major, the mean, eccentric, and true anomalies, while we denote the distance between the planets by $\triangle$. The potential function Ω is then given by the double definite integral

$$\Omega = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{mm'}{\triangle}\,dg\,dg',$$

or, if the integration is accomplished with reference to the eccentric anomalies, by the double definite integral

$$\Omega = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{r}{a}\frac{r'}{a'}\frac{mm'}{\triangle}\,du\,du'.$$

These formulæ show that the potential function is proportional to the average value of the reciprocal of the distance when the mean anomalies are regarded as the independent variables, or to the average value of the product of the radii divided by the distance when the eccentric anomalies are the independent variables. As the eccentricities e and e' and the longitudes of the perihelia ϖ and ϖ' are the variable quantities whose forms as functions of the time we are seeking, it is plain they must be left indeterminate in the expression we obtain for Ω. Since $\triangle$ can be expressed in terms of u and u' as a finite form, the second formula for Ω is to be preferred.

If γ be put for $\varpi - \varpi'$, the expression for $\triangle$, in the case we treat, is

$$\triangle = r'\left[1 - 2\frac{r}{r'}\cos(f - f' + \gamma) + \frac{r^2}{r'^2}\right]^{\frac{1}{2}}.$$

Thus, the expression for Ω becomes

$$\Omega = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{r}{aa'}\frac{mm'}{\left[1 - 2\frac{r}{r'}\cos(f - f' + \gamma) + \frac{r^2}{r'^2}\right]^{\frac{1}{2}}}\,du\,du'.$$

If B_j denote the same function of $\frac{r}{r'}$ that Laplace's $b_{\frac{1}{2}}^{(j)}$ is of a, the ratio of the mean distances, we may write

$$\left[1 - 2\frac{r}{r'}\cos(f - f' + \gamma) + \frac{r^2}{r'^2}\right]^{-\frac{1}{2}} = \frac{1}{2}\sum_{j=-\infty}^{j=+\infty} B_j \cos j(f - f' + \gamma)$$
$$= \frac{1}{2}\sum_{j=-\infty}^{j=+\infty} B_j \varepsilon^{j(f-f'+\gamma)\sqrt{-1}},$$

ε denoting the base of natural logarithms. If we make $\varepsilon^{u\sqrt{-1}} = s$, and put

$$\eta = \frac{1+\sqrt{1-e^2}}{2}, \qquad \omega = \frac{e}{1+\sqrt{1-e^2}},$$

from the equations

$$r = a(1 - e\cos u), \qquad r\cos f = a(\cos u - e), \qquad r\sin f = a\sqrt{1-e^2}\sin u,$$

it is easy to derive

$$r = a\eta(1 - \omega s)\left[1 - \frac{\omega}{s}\right],$$

$$\varepsilon^{f\sqrt{-1}} = \frac{s - \omega}{1 - \omega s}.$$

Thus

$$\frac{r'}{\triangle} = \frac{1}{2}\sum_{j=-\infty}^{j=+\infty} B_j \left[\frac{s-\omega}{1-\omega s}\right]^j \left[\frac{s'-\omega'}{1-\omega' s'}\right]^{-j} \varepsilon^{j\gamma\sqrt{-1}}.$$

Seeking now an expression for B_j in terms of s and s', we have

$$(1 - 2a\cos\varphi + a^2)^{-\frac{1}{2}} = \frac{1}{2}\sum_{j=-\infty}^{j=+\infty} b^{(j)}\varepsilon^{j\phi\sqrt{-1}},$$

(we omit Laplace's subscript $\frac{1}{2}$, as it is unnecessary for the purposes of distinction). We can regard $b^{(j)}$ as an approximate value of B_j, and the true value can be developed in a convergent series by Maclaurin's Theorem, if the perihelion radius of the exterior planet always exceeds the aphelion radius of the interior; that is, if

$$\frac{a'e' + ae}{a' - a} < 1.$$

The augmentation which a receives is

$$\frac{r}{r'} - a = a \frac{\eta (1 - \omega s) \left[1 - \frac{\omega}{s}\right]}{\eta' (1 - \omega' s') \left[1 - \frac{\omega'}{s'}\right]} - a\,.$$

Thus

$$B_j = \sum_{i=0}^{i=+\infty} \frac{1}{i!} a^i \frac{d^i b^{(j)}}{da^i} \left[\frac{\eta (1 - \omega s) \left[1 - \frac{\omega}{s}\right]}{\eta' (1 - \omega' s') \left[1 - \frac{\omega'}{s'}\right]} - 1 \right]^i .$$

Expanding the latter factor by the binomial theorem,

$$B_j = \sum_{i=0}^{i=+\infty} \sum_{k=0}^{k=i} \frac{(-1)^{i-k}}{k!\,(i-k)!} a^i \frac{d^i b^{(j)}}{da^i} \left[\frac{\eta (1 - \omega s) \left[1 - \frac{\omega}{s}\right]}{\eta' (1 - \omega' s') \left[1 - \frac{\omega'}{s'}\right]} \right]^k .$$

Substituting this value of B_j in the expression given above for $\frac{r'}{\triangle}$, and multiplying the result by

$$\frac{mm'r}{aa'} = \frac{mm'}{a'} \eta (1 - \omega s) \left[1 - \frac{\omega}{s}\right],$$

and employing the symbol $\bigtriangledown$ to denote the operation of taking the coefficient of $s^0 s'^0$ in the development of a function of s and s' in a series of integral powers and products of s and s', we shall have

$$\Omega = \frac{mm'}{2a'} \sum_{j=-\infty}^{j=+\infty} \sum_{i=0}^{i=+\infty} \sum_{k=0}^{k=i} \frac{(-1)^{i-k}}{k!\,(i-k)!} a^i \frac{d^i b^{(j)}}{da^i} \eta^{k+1} \eta'^{-k} \varepsilon^{j\gamma\sqrt{-1}}$$
$$\times \bigtriangledown \left[s^j s'^{-j} (1 - \omega s)^{k-j+1} \left[1 - \frac{\omega}{s}\right]^{k+j+1} (1 - \omega' s')^{j-k} \left[1 - \frac{\omega'}{s'}\right]^{-k-j} \right].$$

Let us put

$$E_i^{(j)} = \eta^i \bigtriangledown \left[s^j (1 - \omega s)^{i-j} \left[1 - \frac{\omega}{s}\right]^{i+j} \right].$$

This quantity is then a function of e. Let $E'^{(j)}_i$ be the same function of e' that $E_i^{(j)}$ is of e. Then we can write

$$\Omega = \frac{mm'}{2a'} \sum_{j=-\infty}^{j=+\infty} \sum_{i=0}^{i=+\infty} \sum_{k=0}^{k=i} \frac{(-1)^{i-k}}{k!\,(i-k)!} a^i \frac{d^i b^{(j)}}{da^i} E_{k+1}^{(j)} E'^{(j)}_{-k} \varepsilon^{j\gamma\sqrt{-1}}.$$

This constitutes the infinite series to be employed in this investigation, and it remains only to study the properties of the functions of e denoted by $E_i^{(j)}$. By expanding the binomial factors involved in $E_i^{(j)}$ and performing the operation denoted by ∇, we shall get

$$E_i^{(j)} = (-1)^j \frac{(i+1)(i+2)\ldots(i+j)}{1.2\ldots j} \eta^i \omega^j$$
$$\times\left[1 + \frac{i-j}{1}\,\frac{i}{j+1}\,\omega^2 + \frac{(i-j)(i-j-1)}{1.2}\,\frac{i(i-1)}{(j+1)(j+2)}\,\omega^4 + \ldots\right].$$

The series within the brackets is a case of the hypergeometric series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots,$$

treated by Gauss in a memoir entitled "Disquisitiones generales circa seriem infinitam, etc."* This series gives the value of $E_i^{(j)}$ in terms of η and ω, but it may readily be transformed into another expressed in terms of e. Adopting Gauss's notation for this species of series

$$E_i^{(j)} = (-1)^j \frac{(i+1)(i+2)\ldots(i+j)}{1.2\ldots j} \eta^i \omega^j F(j-i, -i, j+1, \omega^2).$$

But from Gauss's equation [100], p. 225 of the volume quoted,

$$F(j-i, -i, j+1, \omega^2) = (1+\omega^2)^{i-j} F\left[\frac{j-i}{2}, \frac{j-i+1}{2}, j+1, \frac{4\omega^2}{(1+\omega^2)^2}\right],$$

and

$$e^2 = \frac{4\omega^2}{(1+\omega^2)^2}.$$

In consequence

$$E_i^{(j)} = \frac{(i+j)!}{i!\,j!}\left[-\frac{e}{2}\right]^j F\left[\frac{j-i}{2}, \frac{j-i+1}{2}, j+1, e^2\right]$$
$$= \frac{(i+1)\ldots(i+j)}{1\ldots j}\left[-\frac{e}{2}\right]^j\left[1 + \frac{(i-j)(i-j-1)}{1.(j+1)}\left[\frac{e}{2}\right]^2\right.$$
$$\left. + \frac{(i-j)(i-j-1)(i-j-2)(i-j-3)}{1.2.(j+1)(j+2)}\left[\frac{e}{2}\right]^4 + \ldots\right].$$

---

* See Gauss, Werke, Band III, p. 123.

It is remarkable that when i and j are integers the value of $E_i^{(j)}$ is equivalent to a rational function of the two quantities e and $\sqrt{1-e^2}$. For, when i is a positive integer, the series first given terminates after a finite number of terms. The same thing occurs in the second series when $i-j$ is not negative. By Gauss's equation [82], p. 209 of the volume quoted,

$$F\left[\frac{j+i}{2}, \frac{j+i+1}{2}, j+1, e^2\right] = (1-e^2)^{-\frac{2i-1}{2}} F\left[\frac{j-i+2}{2}, \frac{j-i+1}{2}, j+1, e^2\right].$$

From this it follows that

$$\begin{aligned} E_{-i}^{(j)} &= \frac{(i-1)(i-2)\dots(i-j)}{1.2\dots j}\left[\frac{e}{2}\right]^j (1-e^2)^{-\frac{2i-1}{2}} F\left[\frac{j-i+2}{2}, \frac{j-i+1}{2}, j+1, e^2\right] \\ &= \frac{(i-1)\dots(i-j)}{1.2\dots j}\left[\frac{e}{2}\right]^j (1-e^2)^{-\frac{2i-1}{2}} \Bigg[1 + \frac{(i-j-1)(i-j-2)}{1.(j+1)}\left[\frac{e}{2}\right]^2 \\ &\qquad + \frac{(i-j-1)\dots(i-j-4)}{1.2.(j+1)(j+2)}\left[\frac{e}{2}\right]^4 + \dots\Bigg], \end{aligned}$$

which affords a finite expression for $E_i^{(j)}$ when i is negative. It will be noticed that $E_i^{(j)} = 0$, when i, not zero, is not greater than j.

In order that the symmetry of the expression for Ω may be seen, we will write the development of this quantity at length without the employment of the summatory signs:

$$\begin{aligned} \Omega = \frac{mm'}{2a'} \Bigg\{ & b^{(0)} E_1^{(0)} E'^{(0)}_0 \\ & - a\frac{db^{(0)}}{da}\left[E_1^{(0)}E'^{(0)}_0 - E_2^{(0)}E'^{(0)}_{-1}\right] \\ & + \frac{1}{2}a^2\frac{d^2b^{(0)}}{da^2}\left[E_1^{(0)}E'^{(0)}_0 - 2E_2^{(0)}E'^{(0)}_{-1} + E_3^{(0)}E'^{(0)}_{-2}\right] \\ & - \frac{1}{2.3}a^3\frac{d^3b^{(0)}}{da^3}\left[E_1^{(0)}E'^{(0)}_0 - 3E_2^{(0)}E'^{(0)}_{-1} + 3E_3^{(0)}E'^{(0)}_{-2} - E_4^{(0)}E'^{(0)}_{-3}\right] \\ & + \dots\dots\dots\dots\dots\dots \Bigg\} \end{aligned}$$

$$+ \frac{mm'}{a'}\left\{\begin{matrix}\text{Same expression as above, except that } b, E, \text{ and} \\ E' \text{ now take 1 as the upper index instead of 0.}\end{matrix}\right\}\cos\gamma$$

$$+ \frac{mm'}{a'}\left\{\begin{matrix}\text{Same expression, except that } b, E, \text{ and } E' \\ \text{now take 2 as the upper index instead of 0.}\end{matrix}\right\}\cos 2\gamma$$

$$+ \frac{mm'}{a'}\left\{\begin{matrix}\text{Same expression, except that } b, E, \text{ and } E' \\ \text{now take 3 as the upper index instead of 0.}\end{matrix}\right\}\cos 3\gamma$$

$$+ \dots\dots\dots\dots\dots\dots$$

It may be noticed that the terms in $E_2^{(1)}E'^{(1)}_{-1}$, $E_2^{(2)}E'^{(2)}_{-1}$, $E_3^{(2)}E'^{(2)}_{-2}$, $E_2^{(3)}E'^{(3)}_{-1}$, $E_3^{(3)}E'^{(3)}_{-2}$, $E_4^{(3)}E'^{(3)}_{-3}$, etc. can be omitted in writing the expression, as the latter factors of these products vanish. However the symmetry is more apparent when they are retained.

The following table exhibits the values of all the E's required in developing Ω to the terms of the sixth order, inclusive. They are expressed as functions of e, and the finite form is given as perhaps more interesting than the development in ascending powers of e.

$$E_1^{(0)} = 1 \qquad\qquad E_0^{(0)} = 1$$

$$E_2^{(0)} = 1 + \frac{1}{2}e^2 \qquad\qquad E_{-1}^{(0)} = (1 - e^2)^{-\frac{1}{2}}$$

$$E_3^{(0)} = 1 + \frac{3}{2}e^2 \qquad\qquad E_{-2}^{(0)} = (1 - e^2)^{-\frac{3}{2}}$$

$$E_4^{(0)} = 1 + 3e^2 + \frac{3}{8}e^4 \qquad\qquad E_{-3}^{(0)} = \left[1 + \frac{1}{2}e^2\right](1 - e^2)^{-\frac{5}{2}}$$

$$E_5^{(0)} = 1 + 5e^2 + \frac{15}{8}e^4 \qquad\qquad E_{-4}^{(0)} = \left[1 + \frac{3}{2}e^2\right](1 - e^2)^{-\frac{7}{2}}$$

$$E_6^{(0)} = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6 \qquad\qquad E_{-5}^{(0)} = \left[1 + 3e^2 + \frac{3}{8}e^4\right](1 - e^2)^{-\frac{9}{2}}$$

$$E_7^{(0)} = 1 + \frac{21}{2}e^2 + \frac{105}{8}e^4 + \frac{35}{16}e^6 \qquad\qquad E_{-6}^{(0)} = \left[1 + 5e^2 + \frac{15}{8}e^4\right](1 - e^2)^{-\frac{11}{2}}$$

$$E_1^{(1)} = -e \qquad\qquad E_0^{(1)} = -\frac{1 - \sqrt{1 - e^2}}{e}$$

$$E_2^{(1)} = -\frac{3}{2}e \qquad\qquad E_{-1}^{(1)} = 0$$

$$E_3^{(1)} = -2e - \frac{1}{2}e^3 \qquad\qquad E_{-2}^{(1)} = \frac{1}{2}e\,(1 - e^2)^{-\frac{3}{2}}$$

$$E_4^{(1)} = -\frac{5}{2}e - \frac{15}{8}e^3 \qquad\qquad E_{-3}^{(1)} = e\,(1 - e^2)^{-\frac{5}{2}}$$

$$E_5^{(1)} = -3e - \frac{9}{2}e^3 - \frac{3}{8}e^5 \qquad\qquad E_{-4}^{(1)} = \left[\frac{3}{2}e + \frac{3}{8}e^3\right](1 - e^2)^{-\frac{7}{2}}$$

$$E_6^{(1)} = -\frac{7}{2}e - \frac{35}{4}e^3 - \frac{35}{16}e^5 \qquad\qquad E_{-5}^{(1)} = \left[2e + \frac{3}{2}e^3\right](1 - e^2)^{-\frac{9}{2}}$$

$$E_7^{(1)} = -4e - 15e^3 - \frac{15}{2}e^5 - \frac{5}{16}e^7 \qquad\qquad E_{-6}^{(1)} = \left[\frac{5}{2}e + \frac{15}{4}e^3 + \frac{5}{16}e^5\right](1 - e^2)^{-\frac{11}{2}}$$

$$E_1^{(2)} = \frac{3}{2}\left(1 - \sqrt{1-e^2}\right) - \frac{1}{2}\frac{\left(1-\sqrt{1-e^2}\right)^3}{e} \qquad E_0^{(2)} = \frac{\left(1-\sqrt{1-e^2}\right)^2}{e^2}$$

$$E_2^{(2)} = \frac{3}{2}e^2 \qquad E_{-1}^{(2)} = 0$$

$$E_3^{(2)} = \frac{5}{2}e^2 \qquad E_{-2}^{(2)} = 0$$

$$E_4^{(2)} = \frac{15}{4}e^2 + \frac{5}{8}e^4 \qquad E_{-3}^{(2)} = \frac{1}{4}e^2(1-e^2)^{-\frac{5}{2}}$$

$$E_5^{(2)} = \frac{21}{4}e^2 + \frac{21}{8}e^4 \qquad E_{-4}^{(2)} = \frac{3}{4}e^2(1-e^2)^{-\frac{7}{2}}$$

$$E_6^{(2)} = 7e^2 + 7e^4 + \frac{7}{16}e^6 \qquad E_{-5}^{(2)} = \left[\frac{3}{2}e^2 + \frac{1}{4}e^4\right](1-e^2)^{-\frac{9}{2}}$$

$$E_7^{(2)} = 9e^2 + 15e^4 + \frac{45}{16}e^6 \qquad E_{-6}^{(2)} = \left[\frac{5}{2}e^2 + \frac{5}{4}e^4\right](1-e^2)^{-\frac{11}{2}}$$

$$E_1^{(3)} = -2\frac{\left(1-\sqrt{1-e^2}\right)^2}{e} + \frac{\left(1-\sqrt{1-e^2}\right)^4}{e^3} \qquad E_0^{(3)} = -\frac{\left(1-\sqrt{1-e^2}\right)^3}{e^3}$$

$$E_2^{(3)} = -\frac{5}{4}e\left(1-\sqrt{1-e^2}\right) + \frac{5}{4}\frac{\left(1-\sqrt{1-e^2}\right)^3}{e} - \frac{1}{4}\frac{\left(1-\sqrt{1-e^2}\right)^5}{e^3} \qquad E_{-1}^{(3)} = 0$$

$$E_3^{(3)} = -5e^3 \qquad E_{-2}^{(3)} = 0$$

$$E_4^{(3)} = -\frac{35}{4}e^3 \qquad E_{-3}^{(3)} = 0$$

$$E_5^{(3)} = -14e^3 - \frac{7}{4}e^5 \qquad E_{-4}^{(3)} = \frac{1}{8}e^3(1-e^2)^{-\frac{7}{2}}$$

$$E_6^{(3)} = -21e^3 - \frac{63}{8}e^5 \qquad E_{-5}^{(3)} = \frac{1}{2}e^3(1-e^2)^{-\frac{9}{2}}$$

$$E_7^{(3)} = -30e^3 - \frac{45}{2}e^5 - \frac{9}{8}e^7 \qquad E_{-6}^{(3)} = \left[\frac{5}{4}e^3 + \frac{5}{32}e^5\right](1-e^2)^{-\frac{11}{2}}$$

In the present investigation it will be more convenient to make use of a development of $E_i^{(j)}$ in powers of $\sqrt{\left[\frac{1-\sqrt{1-e^2}}{2}\right]} = \theta$. By substituting in the formula for $E_i^{(j)}$ in terms of e the values

$$\left[\frac{e}{2}\right]^2 = \theta^2 - \theta^4,$$

$$\left[\frac{e}{2}\right]^j = \theta^j(1-\theta^2)^{\frac{j}{2}},$$

making, for the sake of brevity, $i - j = k$, and carrying the development to terms of the sixth order, inclusive, we obtain

$$
\begin{aligned}
E_i^{(j)} = (-1)^j \frac{(i+1)\ldots(i+j)}{1\ldots j}\theta^j \Bigg\{ 1 &+ \left[\frac{k(k-1)}{1.(j+1)} - \frac{j}{2}\right]\theta^2 \\
&+ \left[\frac{k(k-1)(k-2)(k-3)}{1.2.(j+1)(j+2)}\right. \\
&\qquad\qquad \left. - \frac{j+2}{2}\,\frac{k(k-1)}{1.(j+1)} + \frac{j(j-2)}{2.4}\right]\theta^4 \\
&+ \left[\frac{k(k-1)(k-2)(k-3)(k-4)(k-5)}{1.2.3.(j+1)(j+2)(j+3)}\right. \\
&\quad - \frac{j+4}{2}\,\frac{k(k-1)(k-2)(k-3)}{1.2.(j+1)(j+2)} \\
&\quad \left. + \frac{j(j+2)}{2.4}\,\frac{k(k-1)}{1.(j+1)} - \frac{j(j-2)(j-4)}{2.4.6}\right]\theta^6 \Bigg\}
\end{aligned}
$$

Or, particularizing with respect to j,

$$
\begin{aligned}
E_i^{(0)} &= 1 + i(i-1)\theta^2 + i(i-1)\left[\frac{(i-2)(i-3)}{2.2} - 1\right]\theta^4 \\
&\qquad + \frac{i(i-1)(i-2)(i-3)}{1.2.1.2}\left[\frac{(i-4)(i-5)}{3.3} - 2\right]\theta^6, \\
E_i^{(1)} &= -(i+1)\theta \Bigg\{ 1 + \left[\frac{(i-1)(i-2)}{1.2} - \frac{1}{2}\right]\theta^2 \\
&\qquad + \left[\frac{(i-1)(i-2)(i-3)(i-4)}{1.2.2.3} - \frac{3}{2}\,\frac{(i-1)(i-2)}{1.2} - \frac{1.1}{2.4}\right]\theta^4 \Bigg\}, \\
E_i^{(2)} &= \frac{(i+1)(i+2)}{1.2}\theta^2 \left\{ 1 + \left[\frac{(i-2)(i-3)}{1.3} - 1\right]\theta^2 \right\}, \\
E_i^{(3)} &= -\frac{(i+1)(i+2)(i+3)}{1.2.3}\theta^3.
\end{aligned}
$$

And, specializing still further,

$$
\begin{array}{ll}
E_1^{(0)} = 1 & E_0^{(0)} = 1 \\
E_2^{(0)} = 1 + 2\theta^2 - 2\theta^4 & E_{-1}^{(0)} = 1 + 2\theta^2 + 4\theta^4 + 8\theta^6 \\
E_3^{(0)} = 1 + 6\theta^2 - 6\theta^4 & E_{-2}^{(0)} = 1 + 6\theta^2 + 24\theta^4 + 80\theta^6 \\
E_4^{(0)} = 1 + 12\theta^2 - 6\theta^4 - 12\theta^6 & E_{-3}^{(0)} = 1 + 12\theta^2 + 78\theta^4 + 380\theta^6 \\
E_5^{(0)} = 1 + 20\theta^2 + 10\theta^4 - 60\theta^6 & E_{-4}^{(0)} = 1 + 20\theta^2 + 190\theta^4 + 1260\theta^6 \\
E_6^{(0)} = 1 + 30\theta^2 + 60\theta^4 - 160\theta^6 & E_{-5}^{(0)} = 1 + 30\theta^2 + 390\theta^4 + 3360\theta^6 \\
E_7^{(0)} = 1 + 42\theta^2 + 168\theta^4 - 280\theta^6 & E_{-6}^{(0)} = 1 + 42\theta^2 + 714\theta^4 + 7728\theta^6
\end{array}
$$

$$E_1^{(1)} = -\theta\left[2 - \theta^2 - \frac{1}{4}\theta^4\right]$$
$$E_3^{(1)} = -\theta\left[4 + 2\theta^2 - \frac{13}{2}\theta^4\right]$$
$$E_4^{(1)} = -\theta\left[5 + \frac{25}{2}\theta^2 - \frac{185}{8}\theta^4\right]$$
$$E_5^{(1)} = -\theta\left[6 + 33\theta^2 - \frac{171}{4}\theta^4\right]$$
$$E_6^{(1)} = -\theta\left[7 + \frac{133}{2}\theta^2 - \frac{287}{8}\theta^4\right]$$
$$E_7^{(1)} = -\theta\,[8 + 116\theta^2 + 59\theta^4]$$

$$E_0^{(1)} = -\theta\left[1 + \frac{1}{2}\theta^2 + \frac{3}{8}\theta^4\right]$$
$$E_{-2}^{(1)} = \theta\left[1 + \frac{11}{2}\theta^2 + \frac{167}{8}\theta^4\right]$$
$$E_{-3}^{(1)} = \theta\left[2 + 19\theta^2 + \frac{439}{4}\theta^4\right]$$
$$E_{-4}^{(1)} = \theta\left[3 + \frac{87}{2}\theta^2 + \frac{2817}{8}\theta^4\right]$$
$$E_{-5}^{(1)} = \theta\left[4 + 82\theta^2 + \frac{1763}{2}\theta^4\right]$$
$$E_{-6}^{(1)} = \theta\left[5 + \frac{275}{2}\theta^2 + \frac{15115}{8}\theta^4\right]$$

$$E_1^{(2)} = \theta^2\,[3 - \theta^2]$$
$$E_4^{(2)} = \theta^2\,[15 - 5\theta^2]$$
$$E_5^{(2)} = \theta^2\,[21 + 21\theta^2]$$
$$E_6^{(2)} = \theta^2\,[28 + 84\theta^2]$$
$$E_7^{(2)} = \theta^2\,[36 + 204\theta^2]$$

$$E_0^{(2)} = \theta^2\,[1 + \theta^2]$$
$$E_{-3}^{(2)} = \theta^2\,[1 + 9\theta^2]$$
$$E_{-4}^{(2)} = \theta^2\,[3 + 39\theta^2]$$
$$E_{-5}^{(2)} = \theta^2\,[6 + 106\theta^2]$$
$$E_{-6}^{(2)} = \theta^2\,[10 + 230\theta^2]$$

$$E_1^{(3)} = -4\theta^3$$
$$E_5^{(3)} = -56\theta^3$$
$$E_6^{(3)} = -84\theta^3$$
$$E_7^{(3)} = -120\theta^3$$

$$E_0^{(3)} = -\theta^3$$
$$E_{-4}^{(3)} = \theta^3$$
$$E_{-5}^{(3)} = 4\theta^3$$
$$E_{-6}^{(3)} = 10\theta^3$$

Through multiplication we obtain

$$E_1^{(0)}E'^{(0)}_0 = 1$$
$$E_2^{(0)}E'^{(0)}_{-1} = 1 + 2\theta^2 + 2\theta'^2 - 2\theta^4 + 4\theta^2\theta'^2 + 4\theta'^4 + 0\theta^6 - 4\theta^4\theta'^2 + 8\theta^2\theta'^4 + 8\theta'^6$$
$$E_3^{(0)}E'^{(0)}_{-2} = 1 + 6\theta^2 + 6\theta'^2 - 6\theta^4 + 36\theta^2\theta'^2 + 24\theta'^4 + 0\theta^6 - 36\theta^4\theta'^2 + 144\theta^2\theta'^4 + 80\theta'^6$$
$$E_4^{(0)}E'^{(0)}_{-3} = 1 + 12\theta^2 + 12\theta'^2 - 6\theta^4 + 144\theta^2\theta'^2 + 78\theta'^4 - 12\theta^6 - 72\theta^4\theta'^2 + 936\theta^2\theta'^4 + 380\theta'^6$$
$$E_5^{(0)}E'^{(0)}_{-4} = 1 + 20\theta^2 + 20\theta'^2 + 10\theta^4 + 400\theta^2\theta'^2 + 190\theta'^4 - 60\theta^6 + 200\theta^4\theta'^2 + 3800\theta^2\theta'^4 + 1260\theta'^6$$
$$E_6^{(0)}E'^{(0)}_{-5} = 1 + 30\theta^2 + 30\theta'^2 + 60\theta^4 + 900\theta^2\theta'^2 + 390\theta'^4 - 160\theta^6 + 1800\theta^4\theta'^2 + 11700\theta^2\theta'^4 + 3360\theta'^6$$
$$E_7^{(0)}E'^{(0)}_{-6} = 1 + 42\theta^2 + 42\theta'^2 + 168\theta^4 + 1764\theta^2\theta'^2 + 714\theta'^4 - 280\theta^6 + 7056\theta^4\theta'^2 + 29988\theta^2\theta'^4 + 7728\theta'^6$$

$$E_1^{(1)}E'^{(1)}_0 = \theta\theta'\left[2 - \theta^2 + \theta'^2 - \frac{1}{4}\theta^4 - \frac{1}{2}\theta^2\theta'^2 + \frac{3}{4}\theta'^4\right]$$

$$E_3^{(1)}E'^{(1)}_{-2} = \theta\theta'\left[-4 - 2\theta^2 - 22\theta'^2 + \frac{13}{2}\theta^4 - 11\theta^2\theta'^2 - \frac{167}{2}\theta'^4\right]$$

$$E_4^{(1)}E'^{(1)}_{-3} = \theta\theta'\left[-10 - 25\theta^2 - 95\theta'^2 + \frac{185}{4}\theta^4 - \frac{475}{2}\theta^2\theta'^2 - \frac{2195}{4}\theta'^4\right]$$

$$E_5^{(1)}E'^{(1)}_{-4} = \theta\theta'\left[-18 - 99\theta^2 - 261\theta'^2 + \frac{513}{4}\theta^4 - \frac{2871}{2}\theta^2\theta'^2 - \frac{8451}{4}\theta'^4\right]$$

$$E_6^{(1)}E'^{(1)}_{-5} = \theta\theta'\left[-28 - 266\theta^2 - 574\theta'^2 + \frac{287}{2}\theta^4 - 5453\theta^2\theta'^2 - \frac{12341}{2}\theta'^4\right]$$

$$E_7^{(1)}E'^{(1)}_{-6} = \theta\theta'\left[-40 - 580\theta^2 - 1100\theta'^2 - 295\theta^4 - 15950\theta^2\theta'^2 - 15115\theta'^4\right]$$

$$E_1^{(2)}E'^{(2)}_0 = \theta^2\theta'^2\left[3 - \theta^2 + 3\theta'^2\right]$$

$$E_4^{(2)}E'^{(2)}_{-3} = \theta^2\theta'^2\left[15 - 5\theta^2 + 135\theta'^2\right]$$

$$E_5^{(2)}E'^{(2)}_{-4} = \theta^2\theta'^2\left[63 + 63\theta^2 + 819\theta'^2\right]$$

$$E_6^{(2)}E'^{(2)}_{-5} = \theta^2\theta'^2\left[168 + 504\theta^2 + 2968\theta'^2\right]$$

$$E_7^{(2)}E'^{(2)}_{-6} = \theta^2\theta'^2\left[360 + 2040\theta^2 + 8280\theta'^2\right]$$

$$E_1^{(3)}E'^{(3)}_0 = 4\theta^3\theta'^3$$

$$E_5^{(3)}E'^{(3)}_{-4} = -56\theta^3\theta'^3$$

$$E_6^{(3)}E'^{(3)}_{-5} = -336\theta^3\theta'^3$$

$$E_7^{(3)}E'^{(3)}_{-6} = -1200\theta^3\theta'^3$$

If, in the expression for Ω, we call the function of the eccentricities which multiplies $\frac{(-1)^i}{i\,!}\,a^i\,\frac{d^i b^{(j)}}{da^i}$ in the coefficient of cos $j\gamma$, $M_i^{(j)}$, and $\triangle$ denoting the characteristic of finite differences with respect to the variable i, it will be seen that we have

$$\triangle^n M_0^{(j)} = (-1)^n E_{n+1}^{(j)} E'^{(j)}_{-n}.$$

Then the expressions for $M_i^{(j)}$ can be derived by considering the preceding expressions, taken alternately with the positive and negative sign, as the successive differences of these functions with respect to the index i; and it will be advantageous to apply the process separately to each power and product of θ and θ'. The exhibition of this follows:—

Coefficients of $\cos 0\gamma$:

Coefficients of θ^0.

$$\begin{array}{rrrrrrr}
1 &&&&&& \\
& -1 &&&&& \\
0 && +1 &&&& \\
& 0 && -1 &&& \\
0 && 0 && +1 && \\
& 0 && 0 && -1 & \\
0 && 0 && 0 && +1 \\
& 0 && 0 && 0 & \\
0 && 0 && 0 && \\
& 0 && 0 &&& \\
0 && 0 &&&& \\
& 0 &&&&& \\
0 &&&&&&
\end{array}$$

Coefficients of θ^2 and θ'^2.

$$\begin{array}{rrrrrrr}
0 &&&&&& \\
& -2 &&&&& \\
-2 && +6 &&&& \\
& +4 && -12 &&& \\
+2 && -6 && +20 && \\
& -2 && +8 && -30 & \\
0 && +2 && -10 && +42 \\
& 0 && -2 && +12 & \\
0 && 0 && +2 && \\
& 0 && 0 &&& \\
0 && 0 &&&& \\
& 0 &&&&& \\
0 &&&&&&
\end{array}$$

Coefficients of θ^4.

$$\begin{array}{rrrrrrr}
0 &&&&&& \\
& +2 &&&&& \\
+2 && -6 &&&& \\
& -4 && +6 &&& \\
-2 && 0 && +10 && \\
& -4 && +16 && -60 & \\
-6 && +16 && -50 && +168 \\
& +12 && -34 && +108 & \\
+6 && -18 && +58 && \\
& -6 && +24 &&& \\
0 && +6 &&&& \\
& 0 &&&&& \\
0 &&&&&&
\end{array}$$

Coefficients of $\theta^2\theta'^2$.

$$\begin{array}{rrrrrrr}
0 &&&&&& \\
& -4 &&&&& \\
-4 && +36 &&&& \\
& +32 && -144 &&& \\
+28 && -108 && +400 && \\
& -76 && +256 && -900 & \\
-48 && +148 && -500 && +1764 \\
& +72 && -244 && +864 & \\
+24 && -96 && +364 && \\
& -24 && +120 &&& \\
0 && +24 &&&& \\
& 0 &&&&& \\
0 &&&&&&
\end{array}$$

Coefficients of θ'^4.

$$\begin{array}{rrrrrrr}
0 &&&&&& \\
& -4 &&&&& \\
-4 && +24 &&&& \\
& +20 && -78 &&& \\
+16 && -54 && +190 && \\
& -34 && +112 && -390 & \\
-18 && +58 && -200 && +714 \\
& +24 && -88 && +324 & \\
+6 && -30 && +124 && \\
& -6 && +36 &&& \\
0 && +6 &&&& \\
& 0 &&&&& \\
0 &&&&&&
\end{array}$$

Coefficients of θ^6.

0	0	0	+ 12	− 60	+ 160	− 280
0	0	+ 12	− 48	+ 100	− 120	
0	+ 12	− 36	+ 52	− 20		
+ 12	− 24	+ 16	+ 32			
− 12	− 8	+ 48				
− 20	+ 40					
+ 20						

Coefficients of $\theta^4\theta'^2$.

0	+ 4	− 36	+ 72	+ 200	− 1800	+ 7056
+ 4	− 32	+ 36	+ 272	− 1600	+ 5256	
− 28	+ 4	+ 308	− 1328	+ 3656		
− 24	+ 312	− 1020	+ 2328			
+ 288	− 708	+ 1308				
− 420	+ 600					
+ 180						

Coefficients of $\theta^2\theta'^4$.

0	− 8	+ 144	− 936	+ 3800	− 11700	+ 29988
− 8	+ 136	− 792	+ 2864	− 7900	+ 18288	
+ 128	− 656	+ 2072	− 5036	+ 10388		
− 528	+ 1416	− 2964	+ 5352			
+ 888	− 1548	+ 2388				
− 660	+ 840					
+ 180						

Coefficients of θ'^6.

0	− 8	+ 80	− 380	+ 1260	− 3360	+ 7728
− 8	+ 72	− 300	+ 880	− 2100	+ 4368	
+ 64	− 228	+ 580	− 1220	+ 2268		
− 164	+ 352	− 640	+ 1048			
+ 188	− 288	+ 408				
− 100	+ 120					
+ 20						

Coefficients multiplying $\theta\theta' \cos \gamma$:

Coefficients of θ^0.

+ 2						
	0					
+ 2		− 4				
	− 4		+ 10			
− 2		+ 6		− 18		
	+ 2		− 8		+ 28	
0		− 2		+ 10		− 40
	0		+ 2		− 12	
0		0		− 2		
	0		0			
0		0				
	0					
0						

Coefficients of θ^2.

− 1						
	0					
− 1		− 2				
	− 2		+ 25			
− 3		+ 23		− 99		
	+ 21		− 74		+ 266	
+ 18		− 51		+ 167		− 580
	− 30		+ 93		− 314	
− 12		+ 42		− 147		
	+ 12		− 54			
0		− 12				
	0					
0						

Coefficients of θ'^2.

+ 1						
	0					
+ 1		− 22				
	− 22		+ 95			
− 21		+ 73		− 261		
	+ 51		− 166		+ 574	
+ 30		− 93		+ 313		− 1100
	− 42		+ 147		− 526	
− 12		+ 54		− 213		
	+ 12		− 66			
0		− 12				
	0					
0						

Coefficients of θ^4 multiplied by 4.

− 1						
	0					
− 1		+ 26				
	+ 26		− 185			
+ 25		− 159		+ 513		
	− 133		+ 328		− 574	
− 108		+ 169		− 61		− 1180
	+ 36		+ 267		− 1754	
− 72		+ 436		− 1815		
	+ 472		− 1548			
+ 400		− 1112				
	− 640					
− 240						

Coefficients of $\theta^2\theta'^2$ multiplied by 2.

− 1	0	− 22	+ 475	− 2871	+ 10906	− 31900
− 1	− 22	+ 453	− 2396	+ 8035	− 20994	
− 23	+ 431	− 1943	+ 5639	− 12959		
+ 408	− 1512	+ 3696	− 7320			
− 1104	+ 2184	− 3624				
+ 1080	− 1440					
− 360						

Coefficients of θ'^4 multiplied by 4.

+ 3	0	− 334	+ 2195	− 8451	+ 24682	− 60460
+ 3	− 334	+ 1861	− 6256	+ 16231	− 35778	
− 331	+ 1527	− 4395	+ 9975	− 19547		
+ 1196	− 2868	+ 5580	− 9572			
− 1672	+ 2712	− 3992				
+ 1040	− 1280					
− 240						

Coefficients multiplying $\theta^2\theta'^2 \cos 2\gamma$:

Coefficients of θ^0.

+ 3	0	0	− 15	+ 63	− 168	+ 360
+ 3	0	− 15	+ 48	− 105	+ 192	
+ 3	− 15	+ 33	− 57	+ 87		
− 12	+ 18	− 24	+ 30			
+ 6	− 6	+ 6				
0	0					
0						

Coefficients of θ^2.

− 1	0	0	+ 5	+ 63	− 504	+ 2040
− 1	0	+ 5	+ 68	− 441	+ 1536	
− 1	+ 5	+ 73	− 373	+ 1095		
+ 4	+ 78	− 300	+ 722			
+ 82	− 222	+ 422				
− 140	+ 200					
+ 60						

Coefficients of θ'^2.

+ 3	0	0	− 135	+ 819	− 2968	+ 8280
+ 3	0	− 135	+ 684	− 2149	+ 5312	
+ 3	− 135	+ 549	− 1465	+ 3163		
− 132	+ 414	− 916	+ 1698			
+ 282	− 502	+ 782				
− 220	+ 280					
+ 60						

Coefficients of $\theta^3\theta'^3 \cos 3\gamma$.

+ 4	0	0	0	− 56	+ 336	− 1200
+ 4	0	0	− 56	+ 280	− 864	
+ 4	0	− 56	+ 224	− 584		
+ 4	− 56	+ 168	− 360			
− 52	+ 112	− 192				
+ 60	− 80					
− 20						

We can now write the explicit development of Ω as follows:

$$
\begin{aligned}
\frac{a'}{mm'}\Omega = \frac{1}{2}\Big\{ & b^{(0)} \\
& + a\frac{db^{(0)}}{da}[2\theta^2 + 2\theta'^2 - 2\theta^4 + 4\theta^2\theta'^2 + 4\theta'^4 + 0\theta^6 - 4\theta^4\theta'^2 + 8\theta^2\theta'^4 + 8\theta'^6] \\
& + \frac{1}{2}a^2\frac{d^2b^{(0)}}{da^2}[2\theta^2 + 2\theta'^2 - 2\theta^4 + 28\theta^2\theta'^2 + 16\theta'^4 \\
& \qquad + 0\theta^6 - 28\theta^4\theta'^2 + 128\theta^2\theta'^4 + 64\theta'^6] \\
& + \frac{1}{2.3}a^3\frac{d^3b^{(0)}}{da^3}[6\theta^4 + 48\theta^2\theta'^2 + 18\theta'^4 - 12\theta^6 \\
& \qquad + 24\theta^4\theta'^2 + 528\theta^2\theta'^4 + 164\theta'^6] \\
& + \frac{1}{2.3.4}a^4\frac{d^4b^{(0)}}{da^4}[6\theta^4 + 24\theta^2\theta'^2 + 6\theta'^4 - 12\theta^6 \\
& \qquad + 288\theta^4\theta'^2 + 888\theta^2\theta'^4 + 188\theta'^6] \\
& + \frac{1}{2.3.4.5}a^5\frac{d^5b^{(0)}}{da^5}[20\theta^6 + 420\theta^4\theta'^2 + 660\theta^2\theta'^4 + 100\theta'^6] \\
& + \frac{1}{2.3.4.5.6}a^6\frac{d^6b^{(0)}}{da^6}[20\theta^6 + 180\theta^4\theta'^2 + 180\theta^2\theta'^4 + 20\theta'^6]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
+\Big\{&\left[b^{(1)} - a\frac{db^{(1)}}{da}\right]\left[2 - \theta^2 + \theta'^2 - \frac{1}{4}\theta^4 - \frac{1}{2}\theta^2\theta'^2 + \frac{3}{4}\theta'^4\right] \\
&- \frac{1}{2}\alpha^2\frac{d^2b^{(1)}}{d\alpha^2}\left[2 + 3\theta^2 + 21\theta'^2 - \frac{25}{4}\theta^4 + \frac{23}{2}\theta^2\theta'^2 + \frac{331}{4}\theta'^4\right] \\
&- \frac{1}{2.3}\alpha^3\frac{d^3b^{(1)}}{d\alpha^3}[18\theta^2 + 30\theta'^2 - 27\theta^4 + 204\theta^2\theta'^2 + 299\theta'^4] \\
&- \frac{1}{2.3.4}\alpha^4\frac{d^4b^{(1)}}{d\alpha^4}[12\theta^2 + 12\theta'^2 + 18\theta^4 + 552\theta^2\theta'^2 + 418\theta'^4] \\
&- \frac{1}{2.3.4.5}\alpha^5\frac{d^5b^{(1)}}{d\alpha^5}[100\theta^4 + 540\theta^2\theta'^2 + 260\theta'^4] \\
&- \frac{1}{2.3.4.5.6}\alpha^6\frac{d^6b^{(1)}}{d\alpha^6}[60\theta^4 + 180\theta^2\theta'^2 + 60\theta'^4]\Big\}\,\theta\theta'\cos\gamma
\end{aligned}
$$

$$
\begin{aligned}
+\Big\{&\left[b^{(2)} - a\frac{db^{(2)}}{da} + \frac{1}{2}\alpha^2\frac{d^2b^{(2)}}{d\alpha^2}\right][3 - \theta^2 + 3\theta'^2] \\
&+ \frac{1}{2.3}\alpha^3\frac{d^3b^{(2)}}{d\alpha^3}[12 - 4\theta^2 + 132\theta'^2] \\
&+ \frac{1}{2.3.4}\alpha^4\frac{d^4b^{(2)}}{d\alpha^4}[6 - 82\theta^2 + 282\theta'^2] \\
&+ \frac{1}{2.3.4.5}\alpha^5\frac{d^5b^{(2)}}{d\alpha^5}[140\theta^2 + 220\theta'^2] \\
&+ \frac{1}{2.3.4.5.6}\alpha^6\frac{d^6b^{(2)}}{d\alpha^6}[60\theta^2 + 60\theta'^2]\Big\}\,\theta^2\theta'^2\cos 2\gamma
\end{aligned}
$$

$$
\begin{aligned}
+\Big\{&4b^{(3)} - 4a\frac{db^{(3)}}{da} + 2\alpha^2\frac{d^2b^{(3)}}{d\alpha^2} - \frac{2}{3}\alpha^3\frac{d^3b^{(3)}}{d\alpha^3} - \frac{13}{6}\alpha^4\frac{d^4b^{(3)}}{d\alpha^4} \\
&- \frac{1}{2}\alpha^5\frac{d^5b^{(3)}}{d\alpha^5} - \frac{1}{36}\alpha^6\frac{d^6b^{(3)}}{d\alpha^6}\Big\}\,\theta^3\theta'^3\cos 3\gamma.
\end{aligned}
$$

In order to have as few functions of a to deal with as possible, we gather together all the terms having the same powers of θ and θ' as factors. Also it will serve our purposes better to have the development of Ω in powers of $\cos\gamma$ than in cosines of multiples of γ. For convenience in writing we denote $a^i\frac{d^ib^{(j)}}{da^i}$ by (j, i). We then put

$$A_1^{(0)} = (0,1) + \frac{1}{2}(0,2),$$

$$A_2^{(0)} = -(0,1) - \frac{1}{2}(0,2) + \frac{1}{2}(0,3) + \frac{1}{8}(0,4),$$

$$A_3^{(0)} = 2(0,1) + 7(0,2) + 4(0,3) + \frac{1}{2}(0,4) - 3(2,0) + 3(2,1)$$
$$- \frac{3}{2}(2,2) - 2(2,3) - \frac{1}{4}(2,4),$$

$$A_4^{(0)} = 2(0,1) + 4(0,2) + \frac{3}{2}(0,3) + \frac{1}{8}(0,4),$$

$$A_5^{(0)} = -(0,3) - \frac{1}{4}(0,4) + \frac{1}{12}(0,5) + \frac{1}{72}(0,6),$$

$$A_6^{(0)} = -2(0,1) - 7(0,2) + 2(0,3) + 6(0,4) + \frac{7}{4}(0,5) + \frac{1}{8}(0,6)$$
$$+ (2,0) - (2,1) + \frac{1}{2}(2,2) + \frac{2}{3}(2,3) - \frac{41}{12}(2,4) - \frac{7}{6}(2,5) - \frac{1}{12}(2,6),$$

$$A_7^{(0)} = 4(0,1) + 32(0,2) + 44(0,3) + \frac{37}{2}(0,4) + \frac{11}{4}(0,5) + \frac{1}{8}(0,6)$$
$$- 3(2,0) + 3(2,1) - \frac{3}{2}(2,2) - 22(2,3) - \frac{47}{4}(2,4) - \frac{11}{6}(2,5) - \frac{1}{12}(2,6),$$

$$A_8^{(0)} = 4(0,1) + 16(0,2) + \frac{41}{3}(0,3) + \frac{47}{12}(0,4) + \frac{5}{12}(0,5) + \frac{1}{72}(0,6),$$

$$A_0^{(1)} = 2(1,0) - 2(1,1) - (1,2),$$

$$A_1^{(1)} = -(1,0) + (1,1) - \frac{3}{2}(1,2) - 3(1,3) - \frac{1}{2}(1,4),$$

$$A_2^{(1)} = (1,0) - (1,1) - \frac{21}{2}(1,2) - 5(1,3) - \frac{1}{2}(1,4),$$

$$A_3^{(1)} = -\frac{1}{4}(1,0) + \frac{1}{4}(1,1) + \frac{25}{8}(1,2) + \frac{9}{2}(1,3) - \frac{3}{4}(1,4) - \frac{5}{6}(1,5) - \frac{1}{12}(1,6),$$

$$A_4^{(1)} = -\frac{1}{2}(1,0) + \frac{1}{2}(1,1) - \frac{23}{4}(1,2) - 34(1,3) - 23(1,4) - \frac{9}{2}(1,5) - \frac{1}{4}(1,6)$$
$$- 12(3,0) + 12(3,1) - 6(3,2) + 2(3,3) + \frac{13}{2}(3,4) + \frac{3}{2}(3,5) + \frac{1}{12}(3,6),$$

$$A_5^{(1)} = \frac{3}{4}(1,0) - \frac{3}{4}(1,1) - \frac{331}{8}(1,2) - \frac{299}{6}(1,3) - \frac{209}{12}(1,4) - \frac{13}{6}(1,5) - \frac{1}{12}(1,6),$$

$$A_0^{(2)} = 3(2,0) - 3(2,1) + \frac{3}{2}(2,2) + 2(2,3) + \frac{1}{4}(2,4),$$

$$A_1^{(2)} = -(2,0) + (2,1) - \frac{1}{2}(2,2) - \frac{2}{3}(2,3) + \frac{41}{12}(2,4) + \frac{7}{6}(2,5) + \frac{1}{12}(2,6),$$

$$A_2^{(2)} = 3(2,0) - 3(2,1) + \frac{3}{2}(2,2) + 22(2,3) + \frac{47}{4}(2,4) + \frac{11}{6}(2,5) + \frac{1}{12}(2,6),$$

$$A_0^{(3)} = 4(3,0) - 4(3,1) + 2(3,2) - \frac{2}{3}(3,3) - \frac{13}{6}(3,4) - \frac{1}{2}(3,5) - \frac{1}{36}(3,6).$$

Then, neglecting the term which is independent of θ, θ', and γ for the reason that it is useless for our purposes, we shall have

$$\begin{aligned}\frac{a'}{mm'}\Omega = {} & A_1^{(0)}(\theta^2+\theta'^2) + A_2^{(0)}\theta^4 + A_3^{(0)}\theta^2\theta'^2 + A_4^{(0)}\theta'^4 + A_5^{(0)}\theta^6 \\ & + A_6^{(0)}\theta^4\theta'^2 + A_7^{(0)}\theta^2\theta'^4 + A_8^{(0)}\theta'^6 \\ & + [A_0^{(1)} + A_1^{(1)}\theta^2 + A_2^{(1)}\theta'^2 + A_3^{(1)}\theta^4 + A_4^{(1)}\theta^2\theta'^2 + A_5^{(1)}\theta'^4]\,\theta\theta'\cos\gamma \\ & + [A_0^{(2)} + A_1^{(2)}\theta^2 + A_2^{(2)}\theta'^2]\,\theta^2\theta'^2\cos^2\gamma \\ & + A_0^{(3)}\theta^3\theta'^3\cos^3\gamma.\end{aligned}$$

In order to make an application of the method to the case of Jupiter and Saturn, we take from Runkle's Tables of the Coefficients of the Perturbative Function the values of log (j, i) corresponding to the argument log $a = 9.7367414$.

i.	$j = 0$.	$j = 1$.	$j = 2$.	$j = 3$.
0	0.3385227	9.7929622	9.4112303	9.0721143
1	9.6447549	9.9080135	9.7803244	9.5982418
2	9.9323686	9.8807530	0.0203420	0.0219693
3	0.2943862	0.3204279	0.3188228	0.3995660
4	0.8737099	0.8712079	0.8884960	0.9011936
5	1.5571487	1.5610571	1.5658243	1.5798073
6	2.3402885	2.3412199	2.3462289	2.3533961

Making use of these values, we obtain for this special case

$$\begin{aligned}\frac{a'}{mm'}\Omega = {} & 0.8692176(\theta^2+\theta'^2) + 1.05019\theta^4 + 11.85269\theta^2\theta'^2 + 8.14486\theta'^4 \\ & + 2.207\theta^6 + 46.126\theta^4\theta'^2 + 157.464\theta^2\theta'^4 + 89.730\theta'^6 \\ & - [1.1365062 + 10.94248\theta^2 + 22.34085\theta'^2 + 42.355\theta^4 \\ & \qquad + 335.261\theta^2\theta'^2 + 362.933\theta'^4]\,\theta\theta'\cos\gamma \\ & + [6.63740 + 86.288\theta^2 + 223.228\theta'^2]\,\theta^2\theta'^2\cos^2\gamma \\ & - 43.209\theta^3\theta'^3\cos^3\gamma.\end{aligned}$$

II.

The portion of the subject which treats of the integration of certain differential equations is now to be attended to. Denoting the mass of the sun by M, and putting

$$\mu = M + m, \quad \mu' = M + m', \quad G = m\sqrt{\mu a}\sqrt{1-e^2}, \quad G' = m'\sqrt{\mu' a'}\sqrt{1-e'^2},$$

the differential equations which determine the eccentricities and positions of the perihelia of the two planets are

$$\frac{dG}{dt}=\frac{d\Omega}{d\tilde{\omega}}, \qquad \frac{d\tilde{\omega}}{dt}=-\frac{d\Omega}{dG},$$

$$\frac{dG'}{dt}=\frac{d\Omega}{d\tilde{\omega}'}, \qquad \frac{d\tilde{\omega}'}{dt}=-\frac{d\Omega}{dG'}.$$

But since Ω involves $\tilde{\omega}$ and $\tilde{\omega}'$ only through $\gamma=\tilde{\omega}-\tilde{\omega}'$, we have

$$\frac{d\Omega}{d\tilde{\omega}}+\frac{d\Omega}{d\tilde{\omega}'}=0.$$

Hence

$$G+G'=\text{a constant}$$

is an integral of the problem. This integral equation may be more suitably expressed in terms of the variables θ and θ' which we have before employed. Then K denoting an arbitrary constant, and denoting the constant quantities $m\sqrt{\mu a}$, $m'\sqrt{\mu' a'}$ by $\frac{1}{\lambda^2}$, $\frac{1}{\lambda'^2}$,

$$\frac{\theta^2}{\lambda^2}+\frac{\theta'^2}{\lambda'^2}=K.$$

The value of K is ascertained by substituting in the left member of this equation for θ and θ' the values they have at a definite epoch. We can now reduce the number of variables in the problem from four to three by adopting a variable ν to replace θ and θ', such that

$$\theta=\lambda\sqrt{K}\sin\tfrac{1}{2}\nu, \qquad \theta'=\lambda'\sqrt{K}\cos\tfrac{1}{2}\nu.$$

$\frac{1}{2}\nu$ remains always in the first quadrant. Denoting the angles of the eccentricities by φ and φ', the eccentricities are determined by the formulæ

$$e=\sin\varphi, \qquad e'=\sin\varphi',$$

$$\sin\tfrac{1}{2}\varphi=\lambda\sqrt{K}\sin\tfrac{1}{2}\nu, \qquad \sin\tfrac{1}{2}\varphi'=\lambda'\sqrt{K}\cos\tfrac{1}{2}\nu.$$

Making the substitutions in Ω necessary to make it involve ν instead of θ and θ', we put

$$\theta^2=\tfrac{1}{2}\lambda^2K(1-\cos\nu), \qquad \theta'^2=\tfrac{1}{2}\lambda'^2K(1+\cos\nu), \qquad \theta\theta'=\tfrac{1}{2}\lambda\lambda'K\sin\nu.$$

The function Ω becomes, then, divisible by K, and, in order to simplify, we shall put $\Omega = KR$. Therefore, if we write x for $\cos \nu$ and put

$$
\begin{aligned}
B_0^{(0)} &= \frac{mm'}{a'} \Big\{ \frac{1}{2}(\lambda^2 + \lambda'^2) A_1^{(0)} + \frac{1}{4}(\lambda^4 A_2^{(0)} + \lambda^2\lambda'^2 A_3^{(0)} + \lambda'^4 A_4^{(0)}) K \\
&\qquad + \frac{1}{8}(\lambda^6 A_5^{(0)} + \lambda^4\lambda'^2 A_6^{(0)} + \lambda^2\lambda'^4 A_7^{(0)} + \lambda'^6 A_8^{(0)}) K^2 \Big\}, \\
B_1^{(0)} &= \frac{mm'}{a'} \Big\{ \frac{1}{2}(-\lambda^2 + \lambda'^2) A_1^{(0)} - \frac{1}{2}(\lambda^4 A_2^{(0)} - \lambda'^4 A_4^{(0)}) K \\
&\qquad + \frac{1}{8}(-3\lambda^6 A_5^{(0)} + \lambda^4\lambda'^2 A_6^{(0)} + \lambda^2\lambda'^4 A_7^{(0)} + 3\lambda'^6 A_8^{(0)}) K^2 \Big\}, \\
B_2^{(0)} &= \frac{mm'}{a'} \Big\{ \frac{1}{4}(\lambda^4 A_2^{(0)} - \lambda^2\lambda'^2 A_3^{(0)} + \lambda'^4 A_4^{(0)}) K \\
&\qquad + \frac{1}{8}(3\lambda^6 A_5^{(0)} - \lambda^4\lambda'^2 A_6^{(0)} - \lambda^2\lambda'^4 A_7^{(0)} + 3\lambda'^6 A_8^{(0)}) K^2 \Big\}, \\
B_3^{(0)} &= \frac{mm'}{a'} \Big\{ \frac{1}{8}(-\lambda^6 A_5^{(0)} + \lambda^4\lambda'^2 A_6^{(0)} - \lambda^2\lambda'^4 A_7^{(0)} + \lambda'^6 A_8^{(0)}) K^2 \Big\}, \\
B_0^{(1)} &= \frac{mm'}{a'} \frac{\lambda\lambda'}{2} \Big\{ A_0^{(1)} + \frac{1}{2}(\lambda^2 A_1^{(1)} + \lambda'^2 A_2^{(1)}) K + \frac{1}{4}(\lambda^4 A_3^{(1)} + \lambda^2\lambda'^2 A_4^{(1)} + \lambda'^4 A_5^{(1)}) K^2 \Big\}, \\
B_1^{(1)} &= \frac{mm'}{a'} \frac{\lambda\lambda'}{2} \Big\{ \frac{1}{2}(-\lambda^2 A_1^{(1)} + \lambda'^2 A_2^{(1)}) K + \frac{1}{2}(-\lambda^4 A_3^{(1)} + \lambda'^4 A_5^{(1)}) K^2 \Big\}, \\
B_2^{(1)} &= \frac{mm'}{a'} \frac{\lambda\lambda'}{2} \Big\{ \frac{1}{4}(\lambda^4 A_3^{(1)} - \lambda^2\lambda'^2 A_4^{(1)} + \lambda'^4 A_5^{(1)}) K^2 \Big\}, \\
B_0^{(2)} &= \frac{mm'}{a'} \frac{\lambda^2\lambda'^2}{4} \Big\{ A_0^{(2)} K + \frac{1}{2}(\lambda^2 A_1^{(2)} + \lambda'^2 A_2^{(2)}) K^2 \Big\}, \\
B_1^{(2)} &= \frac{mm'}{a'} \frac{\lambda^2\lambda'^2}{8} (-\lambda^2 A_1^{(2)} + \lambda'^2 A_2^{(2)}) K^2, \\
B_0^{(3)} &= \frac{mm'}{a'} \frac{\lambda^3\lambda'^3}{8} A_0^{(3)} K^2;
\end{aligned}
$$

we shall then have

$$
\begin{aligned}
R = {} & B_0^{(0)} + B_1^{(0)} x + B_2^{(0)} x^2 + B_3^{(0)} x^3 + \ldots \\
& + [B_0^{(1)} + B_1^{(1)} x + B_2^{(1)} x^2 + \ldots] \sin \nu \cos \gamma \\
& + [B_0^{(2)} + B_1^{(2)} x + \ldots] \sin^2 \nu \cos^2 \gamma \\
& + [B_0^{(3)} + \ldots] \sin^3 \nu \cos^3 \gamma \\
& + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned}
$$

With this expression for R it is readily seen from the preceding differential equations that the differential equation determining ν is

$$\frac{d\nu}{dt} = -\frac{1}{\sin\nu}\frac{dR}{d\gamma},$$

or

$$\frac{dx}{dt} = \frac{dR}{d\gamma}.$$

Since $R =$ a constant is evidently an integral of the problem, we shall have

$$\frac{dR}{d\nu}\frac{d\nu}{dt} + \frac{dR}{d\gamma}\frac{d\gamma}{dt} = 0.$$

Whence is derived

$$\frac{d\gamma}{dt} = \frac{1}{\sin\nu}\frac{dR}{d\nu}.$$

We still need an additional equation giving the value of some other function of $\tilde{\omega}$ and $\tilde{\omega}'$ than $\tilde{\omega} - \tilde{\omega}'$. If we select $\tilde{\omega} + \tilde{\omega}'$ we have

$$\frac{d(\tilde{\omega} + \tilde{\omega}')}{dt} = -\frac{d\Omega}{dG} - \frac{d\Omega}{dG'}.$$

If K is kept evident in the expressions for the various B's, so that the partial derivatives of them with respect to this quantity may be taken, we shall have

$$\frac{d\Omega}{dG} = \frac{d(KR)}{dK}\frac{dK}{dG} + K\frac{dR}{d\nu}\frac{d\nu}{dG} = -\frac{1}{2}\frac{d(KR)}{dK} - \frac{1}{2\tan\frac{\nu}{2}}\frac{dR}{d\nu},$$

$$\frac{d\Omega}{dG'} = \frac{d(KR)}{dK}\frac{dK}{dG'} + K\frac{dR}{d\nu}\frac{d\nu}{dG'} = -\frac{1}{2}\frac{d(KR)}{dK} + \frac{1}{2}\tan\frac{\nu}{2}\frac{dR}{d\nu}.$$

Whence

$$\begin{aligned}\frac{d(\tilde{\omega} + \tilde{\omega}')}{dt} &= \frac{d(KR)}{dK} + \frac{\cos\nu}{\sin\nu}\frac{dR}{d\nu} \\ &= \frac{d(KR)}{dK} + \cos\nu\frac{d\gamma}{dt}.\end{aligned}$$

In making our numerical application we take the mean distance a' as the

unit, when a becomes the same as a previously given, and assume for the masses the values

$$m = \frac{1}{1047.879}, \qquad m' = \frac{1}{3482.2}.$$

These give

$$\log \lambda = 1.5758667, \qquad \log \lambda' = 1.7708956.$$

The values adopted for the eccentricities at the beginning of 1850 are

$$e = 0.04825801, \qquad e' = 0.05606467.$$

These furnish the equations

$$\theta = [8.4778154] \sin \frac{\nu}{2}, \qquad \theta' = [8.6728444] \cos \frac{\nu}{2},$$

and the function R becomes

$$\begin{aligned} R = {} & 0.0005906465 + 0.0002543964x + 0.00000196780x^2 \\ & + 0.000000019394x^3 \\ & - [0.0003548741 + 0.00000629406x + 0.00000008731x^2] \sin \nu \cos \gamma \\ & + [0.00000148778 + 0.00000004479x] \sin^2 \nu \cos^2 \gamma \\ & - 0.000000006560 \sin^3 \nu \cos^3 \gamma. \end{aligned}$$

The value of the constant in the integral equation

$$R = C$$

is ascertained by substituting in the expression for R the values which ν and γ have at a definite epoch, as 1850. C being determined, the equation $R = C$ can be solved, regarding $\sin \nu \cos \gamma$ as the quantity whose value is to be obtained. This value can be supposed developed in powers of $\cos \nu = x$, and we write

$$\sin \nu \cos \gamma = H = D_0 + D_1 x + D_2 x^2 + D_3 x^3 + \dots .$$

The readiest method of obtaining the D's is by substituting the last expression in R and then equating the resulting coefficients of each power of x to zero. We thus have a system of equations determining the D's. These can be solved by successive approximation. If C is allowed to appear as an indeterminate in the expressions for the D's, H can be partially differentiated with reference to this quantity.

We can now make H play the rôle of R; for we have

$$\frac{dx}{dt}=\frac{dR}{d\gamma}, \qquad \frac{d\gamma}{dt}=-\frac{dR}{dx}, \qquad \text{and} \qquad \gamma=\text{arc cos}\frac{H}{\sqrt{1-x^2}}.$$

Thus

$$\frac{dt}{dx}=\frac{d\gamma}{dC}=-\frac{\frac{dH}{dC}}{\sqrt{1-x^2-H^2}},$$

where the radical in the denominator must receive the sign of $\sin\gamma$; for we have

$$\begin{aligned}\cos\nu&=x,\\ \sin\nu\cos\gamma&=H=D_0+D_1x+D_2x^2+D_3x^3+\dots,\\ \sin\nu\sin\gamma&=\sqrt{(1-x^2-H^2)}.\end{aligned}$$

If we suppose the orbits are always ellipses x cannot pass the limits ± 1. Thus x must oscillate between a maximum and a minimum value, while dH/dC remains constantly of the same sign. The maximum and minimum values of x are evidently the two consecutive real roots of the equation in x

$$1-x^2-H^2=0,$$

which contain between them at any time the actual value of x. Calling these roots a and b, we may write

$$1-x^2-H^2=(\mathrm{a}-x)(x-\mathrm{b})Q,$$

where Q is positive for all values of x lying between a and b; and when the eccentricities are always small, the variation of Q is slight in comparison with its magnitude. In the place of x we can adopt a new variable, ψ, such that

$$x=\frac{\mathrm{a}+\mathrm{b}}{2}-\frac{\mathrm{a}-\mathrm{b}}{2}\cos\psi.$$

Then

$$\frac{dx}{\sqrt{(\mathrm{a}-x)(x-\mathrm{b})}}=d\psi,$$

and the differential equation giving ψ in terms of t is

$$\frac{dt}{d\psi}=-\frac{\frac{dH}{dC}}{\sqrt{Q}}.$$

To see how all this applies in the case of Jupiter and Saturn we assume the following values of the longitudes of the perihelia at the epoch 1850.0:

$$\tilde{\omega} = 11^\circ\ 54'\ 31''.18, \qquad \tilde{\omega}' = 90^\circ\ 6'\ 57''.55.$$

The value of the constant C being now determined, and the equation $R = C$ modified in such a way that it becomes more suitable for solution, we have

$$\begin{aligned} 0.4021256 = &- 0.7168638x - 0.0055451x^2 - 0.0000546x^3 \\ &+ [1 + 0.0177360x + 0.0002460x^2] \sin \nu \cos \gamma \\ &- [0.0041924 + 0.0001262x] \sin^2\nu \cos^2\gamma \\ &+ 0.0000185 \sin^3\nu \cos^3\gamma. \end{aligned}$$

When this equation is solved with reference to $\sin\nu \cos\gamma$ as the unknown, we obtain

$$H = 0.4028046 + 0.7121389x - 0.0050141x^2 - 0.0000050x^3.$$

And when we ascertain what increment H receives from an infinitesimal increment in the quantity C, it results that

$$-\frac{dH}{dC} = 2827.425 - 33.179x + 0.005x^2.$$

The equation $1 - x^2 - H^2 = 0$, in this case, is

$$0.8377485 - 0.5737057x - 1.5031028x^2 + 0.0071447x^3 - 0.0000180x^4 = 0.$$

The consecutive real roots of this which contain between them the value of x at 1850.0 are

$$\mathrm{a} = 0.5803236, \qquad \mathrm{b} = -0.9586738.$$

We derive from these the limiting values of ν, which are

$$54^\circ\ 31'\ 36''.14 \qquad \text{and} \qquad 163^\circ\ 28'\ 14''.01.$$

Thus, when $\gamma = 0^\circ$, the minimum e of Jupiter has place, which is 0.02752623; as also the maximum e' of Saturn, which is 0.08362800. And, when $\gamma = 180^\circ$, the maximum e of Jupiter has place, and is 0.05944555; and the minimum e' of Saturn, which is 0.01353514.

The remaining factor of the equation, two of whose roots we have just obtained, is

$$Q = 1.5058180 - 0.0071522x + 0.0000180x^2.$$

Whence

$$\frac{1}{\sqrt{Q}} = 0.8149177 + 0.0019353x + 0.0000020x^2.$$

Substituting, then, for x the expression

$$x = -0.1891751 - 0.7694987 \cos\psi,$$

we get

$$\begin{aligned}\frac{dt}{d\psi} &= 2304.1185 - 21.5662x - 0.0543x^2 \\ &= 2308.1802 + 16.5794 \cos\psi - 0.0161 \cos 2\psi.\end{aligned}$$

Integrating this, c being the arbitrary constant,

$$t + c = 2308.1802\psi + 16.5794 \sin\psi - 0.0080 \sin 2\psi.$$

Inverting this series and changing the numerical coefficients into seconds of arc we get

$$\begin{aligned}\psi = 19''.05825\,(t + c) &- 1481''.57 \sin[19''.05825\,(t + c)] \\ &+ 6''.04 \sin 2\,[19''.05825\,(t + c)].\end{aligned}$$

From the value which ψ must have at the epoch 1850.0, t being counted thence,

$$19''.05825c = 277^\circ\ 9'\ 9''.15.$$

Also, we have

$$\begin{aligned}\cos\nu &= -0.1891751 - 0.7694987 \cos\psi, \\ \sin\nu\cos\gamma &= +0.2679063 - 0.5494490 \cos\psi - 0.0029673 \cos^2\psi \\ &\quad + 0.0000023 \cos^3\psi, \\ \sin\nu\sin\gamma &= [0.9446898 + 0.0017265 \cos\psi + 0.0000018 \cos^2\psi] \sin\psi.\end{aligned}$$

These equations enable us to determine the eccentricities and difference of the longitudes of the perihelia at any given time.

It remains to find the longitudes of the perihelia themselves. We have

$$\frac{d\tilde{\omega}}{dt} = \tfrac{1}{2} C + \frac{1}{2} K \frac{dR}{dK} + \frac{1+x}{2} \frac{d\gamma}{dt},$$

$$\frac{d\tilde{\omega}'}{dt} = \tfrac{1}{2} C + \frac{1}{2} K \frac{dR}{dK} - \frac{1-x}{2} \frac{d\gamma}{dt}.$$

Or

$$\frac{d(\tilde{\omega} - \tfrac{1}{2} Ct)}{dx} = -\frac{1}{2} K \frac{d\gamma}{dK} + \frac{1+x}{2} \frac{d\gamma}{dx},$$

$$\frac{d(\tilde{\omega}' - \tfrac{1}{2} Ct)}{dx} = -\frac{1}{2} K \frac{d\gamma}{dK} - \frac{1-x}{2} \frac{d\gamma}{dx}.$$

Or

$$\frac{d(\tilde{\omega} - \tfrac{1}{2} Ct)}{dx} = \frac{1}{2} \frac{K \dfrac{dH}{dK} - (1+x) \dfrac{dH}{dx} - \dfrac{Hx}{1-x}}{\sqrt{(1 - x^2 - H^2)}},$$

$$\frac{d(\tilde{\omega}' - \tfrac{1}{2} Ct)}{dx} = \frac{1}{2} \frac{K \dfrac{dH}{dK} + (1-x) \dfrac{dH}{dx} + \dfrac{Hx}{1+x}}{\sqrt{(1 - x^2 - H^2)}}.$$

Here K must be left indeterminate in the coefficients D_0, D_1, etc. of H, in order that we may get $\dfrac{dH}{dK}$. In the next place, we derive

$$\frac{d(\tilde{\omega} - \tfrac{1}{2} Ct)}{d\psi} = \frac{1}{2} \frac{K \dfrac{dH}{dK} - (1+x) \dfrac{dH}{dx} - \dfrac{Hx}{1-x}}{\sqrt{Q}},$$

$$\frac{d(\tilde{\omega}' - \tfrac{1}{2} Ct)}{d\psi} = \frac{1}{2} \frac{K \dfrac{dH}{dK} + (1-x) \dfrac{dH}{dx} + \dfrac{Hx}{1+x}}{\sqrt{Q}}.$$

When H, which is an infinite series in integral powers of x, is divided by $1 - x$ or $1 + x$, remainders independent of x are left over which are equivalent to what H becomes when in it we make $x = 1$ and $x = -1$. These remainders we denote as $H(1)$ and $H(-1)$. Then we may write

$$\frac{d(\tilde{\omega} - \tfrac{1}{2} Ct)}{d\psi} = \frac{1}{2} \frac{-\dfrac{H(1)}{1-x} + \sum_{i=0}^{i=\infty}\left[K \dfrac{dD_i}{dK} - (i-1)D_i - iD_{i+1} + D_{i+2} + D_{i+3} + \ldots\right]x^i}{\sqrt{Q}},$$

$$\frac{d(\tilde{\omega}' - \tfrac{1}{2} Ct)}{d\psi} = \frac{1}{2} \frac{-\dfrac{H(-1)}{1+x} + \sum_{i=0}^{i=\infty}\left[K \dfrac{dD_i}{dK} - (i-1)D_i + iD_{i+1} + D_{i+2} - D_{i+3} + \ldots\right]x^i}{\sqrt{Q}}.$$

The difference of these equations gives

$$\frac{d\gamma}{d\psi} = \frac{-\frac{1}{2}\frac{H(1)}{1-x} + \frac{1}{2}\frac{H(-1)}{1+x} + \sum\limits_{i=0}^{i=\infty}\Big[-iD_{i+1} + D_{i+3} + D_{i+5}\Big]x^i}{\sqrt{Q}}.$$

Since γ returns to the same value after ψ has augmented by a circumference, it follows that when the right member is expanded in an infinite series containing, besides two terms in the form of fractions having $1-x$ and $1+x$ as denominators, a set of terms proceeding according to cosines of multiples of ψ, the coefficient of the zero multiple of ψ must vanish. This is not immediately evident from the form of the expression. Hence I proceed to prove it to the degree of approximation we adopt. Let

$$\frac{1}{\sqrt{Q}} = E_0 + E_1 x + E_2 x^2 + \ldots;$$

then, omitting the two terms in the form of fractions and having $1-x$ and $1+x$ for denominators, it will be perceived that we have

$$\frac{d\gamma}{d\psi} = D_3E_0 + (D_0 + D_2)E_1 + D_1E_2 - [D_2E_0 - D_0E_2]\,x - [2D_3E_0 - D_2E_1]\,x^2.$$

Substituting for x its value in terms of ψ, if our proposition is true we ought to have

$$D_3E_0 + (D_0 + D_2)E_1 + D_1E_2 - [D_2E_0 - D_0E_2]\frac{\mathrm{a}+\mathrm{b}}{2}$$
$$- [2D_3E_0 - D_2E_1]\left[\frac{3}{2}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2 - \frac{1}{2}\mathrm{ab}\right] = 0.$$

But if

$$Q = M_0 + M_1x + M_2x^2 + \ldots,$$

$$E_0 = M_0^{-\frac{1}{2}}, \quad E_1 = -\frac{1}{2}M_0^{-\frac{3}{2}}M_1, \quad E_2 = -\frac{1}{2}M_0^{-\frac{3}{2}}M_2 + \frac{3}{8}M_0^{-\frac{5}{2}}M_1^2,$$

and M_0, M_1, M_2, a, and b are determined by the equations

$$\mathrm{ab}M_0 = D_0^2 - 1,$$
$$(\mathrm{a}+\mathrm{b})M_0 - \mathrm{ab}M_1 = -2D_0D_1,$$
$$M_0 - (\mathrm{a}+\mathrm{b})M_1 + \mathrm{ab}M_2 = 1 + D_1^2 + 2D_0D_2,$$
$$M_1 - (\mathrm{a}+\mathrm{b})M_2 = 2(D_1D_2 + D_0D_3),$$
$$M_2 = D_2^2 + 2D_1D_3.$$

By substituting the values of E_0, E_1, and E_2 and multiplying by M_0^3, our equation becomes

$$\left\{-D_2\frac{\mathrm{a}+\mathrm{b}}{2}+D_3\left[1-3\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2+\mathrm{ab}\right]\right\}M_0$$

$$-\frac{1}{2}\left\{D_0+D_2\left[1-\frac{3}{2}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2+\frac{1}{2}\mathrm{ab}\right]\right\}M_1$$

$$+\left[D_1+D_0\frac{\mathrm{a}+\mathrm{b}}{2}\right]\left[-\frac{1}{2}M_2+\frac{3}{8}\frac{M_1^2}{M_0}\right]=0.$$

But

$$\frac{\mathrm{a}+\mathrm{b}}{2}M_0=-D_0D_1+\frac{1}{2}\mathrm{ab}M_1,$$

$$-D_2\frac{\mathrm{a}+\mathrm{b}}{2}M_0=D_0D_1D_2-\frac{1}{2}D_2\mathrm{ab}M_1,$$

$$\frac{1}{2}M_1=D_1D_2+D_0D_3+\frac{\mathrm{a}+\mathrm{b}}{2}M_2,$$

$$-\frac{1}{2}D_0M_1=-D_0D_1D_2-D_0^2D_3-D_0\frac{\mathrm{a}+\mathrm{b}}{2}M_2.$$

By substituting these, the equation becomes

$$-D_0^2D_3+D_3\left[1-3\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2+\mathrm{ab}\right]M_0-\frac{1}{2}D_2\left[1-\frac{3}{2}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2+\frac{3}{2}\mathrm{ab}M_1\right]$$

$$-\frac{1}{2}D_1\left[M_2-\frac{3}{4}\frac{M_1^2}{M_0}\right]-D_0\frac{\mathrm{a}+\mathrm{b}}{2}\left[\frac{3}{2}M_2-\frac{3}{8}\frac{M_1^2}{M_0}\right]=0.$$

This may easily be transformed into

$$-D_0^2D_3+D_3\left[1+D_1^2+3D_0D_1\frac{\mathrm{a}+\mathrm{b}}{2}+D_0^2-1\right]$$

$$-D_1D_2^2\left[1-\frac{3}{2}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2+\frac{3}{2}\mathrm{ab}\right]$$

$$-\frac{1}{2}D_1\left[D_2^2+2D_1D_3-3\frac{D_1^2D_2^2}{M_0}\right]$$

$$-D_0\frac{\mathrm{a}+\mathrm{b}}{2}\left[\frac{3}{2}D_2^2+3D_1D_3-\frac{3}{2}\frac{D_1^2D_2^2}{M_0}\right]=0.$$

Which reduces to

$$-\left[1+\frac{3}{2}\frac{\mathrm{a}+\mathrm{b}}{2}\frac{D_0D_1}{M_0}+\frac{3}{2}\frac{D_0^2-1}{M_0}\right]D_1D_2^2-\frac{1}{2}D_1\left[D_2^2-3\frac{D_1^2D_2^2}{M_0}\right]$$
$$-D_0\frac{\mathrm{a}+\mathrm{b}}{2}\left[\frac{3}{2}D_2^2-\frac{3}{2}\frac{D_1^2D_2^2}{M_0}\right]=0,$$

and thence to

$$-\left[1+\frac{1}{2}+\frac{3}{2}\frac{D_0^2-1}{D_1^2+1}-\frac{3}{2}\frac{D_1^2}{D_1^2+1}-\frac{3}{2}\frac{D_0^2}{D_1^2+1}\right]D_1D_2^2=0,$$

which is perceived to be identical.

When

$$\frac{1}{\sqrt{Q}}=E_0+E_1x+E_2x^2+\ldots$$

is divided by $1-x$ the remainder is equivalent to what $\frac{1}{\sqrt{Q}}$ becomes when x is put equal to 1. But

$$\frac{1}{\sqrt{Q}}=\sqrt{\frac{(\mathrm{a}-x)(x-\mathrm{b})}{1-x^2-H^2}},$$

consequently this remainder is

$$\pm\frac{\sqrt{(1-\mathrm{a})(1-\mathrm{b})}}{H(1)},$$

the ambiguous sign being so taken as to render the quantity positive. In like manner it is shown that the remainder of $\frac{1}{\sqrt{Q}}$ divided by $1+x$ is

$$\pm\frac{\sqrt{(1+\mathrm{a})(1+\mathrm{b})}}{H(-1)}.$$

Then

$$\frac{d(\tilde{\omega}-\frac{1}{2}Ct)}{d\psi}=\mp\frac{1}{2}\frac{\sqrt{(1-\mathrm{a})(1-\mathrm{b})}}{1-x}+L_0+L_1x+L_2x^2+\ldots,$$

where the upper or lower sign is to be taken according as $H(1)$ is positive or negative. And

$$\frac{d(\tilde{\omega}'-\frac{1}{2}Ct)}{d\psi}=\mp\frac{1}{2}\frac{\sqrt{(1+\mathrm{a})(1+\mathrm{b})}}{1+x}+L_0'+L_1'x+L_2'x^2+\ldots,$$

where the upper or lower sign is to be taken according as $H(-1)$ is positive or negative. The expressions for the L and L', correct to quantities of the order of the fourth power of the eccentricities inclusive, are

$$2L_0 = \left[K\frac{dD_0}{dK} + D_0 + D_2 + D_3\right]E_0 + H(1)\,[E_1 + E_2]\,,$$

$$2L_1 = \left[K\frac{dD_1}{dK} - D_2 + D_3\right]E_0 + \left[K\frac{dD_0}{dK} + D_0 + D_2\right]E_1 + H(1)\,E_2\,,$$

$$2L_2 = \left[K\frac{dD_2}{dK} - D_2 - 2D_3\right]E_0 + \left[K\frac{dD_1}{dK} - D_2\right]E_1 + D_0E_2\,,$$

$$2L_0' = \left[K\frac{dD_0}{dK} + D_0 + D_2 - D_3\right]E_0 - H(-1)\,[E_1 - E_2]\,,$$

$$2L_1' = \left[K\frac{dD_1}{dK} + D_2 + D_3\right]E_0 + \left[K\frac{dD_0}{dK} + D_0 + D_2\right]E_1 - H(-1)\,E_2\,,$$

$$2L_2' = \left[K\frac{dD_2}{dK} - D_2 + 2D_3\right]E_0 + \left[K\frac{dD_1}{dK} + D_2\right]E_1 + D_0E_2\,.$$

By substituting the value

$$x = \frac{\mathrm{a}+\mathrm{b}}{2} - \frac{\mathrm{a}-\mathrm{b}}{2}\cos\psi\,,$$

and putting

$$N_0 = L_0 + L_1\frac{\mathrm{a}+\mathrm{b}}{2} + L_2\left[\frac{3}{2}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2 - \frac{1}{2}\,\mathrm{ab}\right],$$

$$N_1 = -\,L_1\frac{\mathrm{a}-\mathrm{b}}{2} - L_2\frac{\mathrm{a}^2-\mathrm{b}^2}{2}\,,$$

$$N_2 = L_2\frac{(\mathrm{a}-\mathrm{b})^2}{8}\,,$$

$$N'_0 = L_0' + L_1'\frac{\mathrm{a}+\mathrm{b}}{2} + L_2'\left[\frac{3}{2}\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^2 - \frac{1}{2}\,\mathrm{ab}\right],$$

$$N'_1 = -\,L_1'\frac{\mathrm{a}-\mathrm{b}}{2} - L_2'\frac{\mathrm{a}^2-\mathrm{b}^2}{2}\,,$$

$$N'_2 = L_2'\frac{(\mathrm{a}-\mathrm{b})^2}{8}\,,$$

where we have, as has been proved above, the relation $N_0 = N_0'$, we get

$$\frac{d(\varpi - \frac{1}{2}Ct)}{d\psi} = \mp \frac{\frac{1}{2}\sqrt{(1-\mathrm{a})(1-\mathrm{b})}}{1 - \frac{\mathrm{a}+\mathrm{b}}{2} + \frac{\mathrm{a}-\mathrm{b}}{2}\cos\psi} + N_0 + N_1\cos\psi + N_2\cos 2\psi + \ldots,$$

$$\frac{d(\varpi' - \frac{1}{2}Ct)}{d\psi} = \mp \frac{\frac{1}{2}\sqrt{(1+\mathrm{a})(1+\mathrm{b})}}{1 + \frac{\mathrm{a}+\mathrm{b}}{2} - \frac{\mathrm{a}-\mathrm{b}}{2}\cos\psi} + N_0' + N_1'\cos\psi + N_2'\cos 2\psi + \ldots.$$

Integrating, we have

$$\varpi - \tfrac{1}{2}Ct = c \mp \operatorname{arc\,tan}\left[\sqrt{\frac{1-\mathrm{a}}{1-\mathrm{b}}}\tan\frac{\psi}{2}\right] + N_0\psi + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots.$$

$$\varpi' - \tfrac{1}{2}Ct = c' \mp \operatorname{arc\,tan}\left[\sqrt{\frac{1+\mathrm{a}}{1+\mathrm{b}}}\tan\frac{\psi}{2}\right] + N_0'\psi + N_1'\sin\psi + \tfrac{1}{2}N_2'\sin 2\psi + \ldots.$$

The quadrant in which the arc correspondent to the tangent is to be taken is found by dividing the number of the quadrant of ψ by 2, if it is even; or by augmenting the number of the quadrant of ψ by unity, if it is odd, and then dividing by 2.

By taking the sine, we have, β being any arbitrary angle,

$$\sqrt{1-x}\sin(\varpi - \tfrac{1}{2}Ct + \beta)$$

$$= \mp\sqrt{1-\mathrm{a}}\sin\frac{\psi}{2}\cos[N_0\psi + c + \beta + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots]$$

$$+ \sqrt{1-\mathrm{b}}\cos\frac{\psi}{2}\sin[N_0\psi + c + \beta + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots],$$

$$\sqrt{1+x}\sin(\varpi' - \tfrac{1}{2}Ct + \beta)$$

$$= \mp\sqrt{1+\mathrm{a}}\sin\frac{\psi}{2}\cos[N_0'\psi + c' + \beta + N_1'\sin\psi + \tfrac{1}{2}N_2'\sin 2\psi + \ldots]$$

$$+ \sqrt{1+\mathrm{b}}\cos\frac{\psi}{2}\sin[N_0'\psi + c' + \beta + N_1'\sin\psi + \tfrac{1}{2}N_2'\sin 2\psi + \ldots]$$

or, as they may be written,

$$\sqrt{1-x}\,\sin(\tilde{\omega} - \tfrac{1}{2}Ct + \beta)$$

$$= \tfrac{1}{2}\big[\sqrt{1-\mathrm{b}} \mp \sqrt{1-\mathrm{a}}\big]\sin\big[(N_0+\tfrac{1}{2})\psi + c + \beta + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots\big]$$

$$+ \tfrac{1}{2}\big[\sqrt{1-\mathrm{b}} \pm \sqrt{1-\mathrm{a}}\big]\sin\big[(N_0-\tfrac{1}{2})\psi + c + \beta + N_1\sin\psi + \tfrac{1}{2}N_2\sin 2\psi + \ldots\big],$$

$$\sqrt{1+x}\,\sin(\tilde{\omega}' - \tfrac{1}{2}Ct + \beta)$$

$$= \tfrac{1}{2}\big[\sqrt{1+\mathrm{b}} \mp \sqrt{1+\mathrm{a}}\big]\sin\big[(N_0'+\tfrac{1}{2})\psi + c' + \beta + N_1'\sin\psi + \tfrac{1}{2}N_2'\sin 2\psi + \ldots\big]$$

$$+ \tfrac{1}{2}\big[\sqrt{1+\mathrm{b}} \pm \sqrt{1+\mathrm{a}}\big]\sin\big[(N_0'-\tfrac{1}{2})\psi + c' + \beta + N_1'\sin\psi + \tfrac{1}{2}N_2'\sin 2\psi + \ldots\big].$$

The expression for the auxiliary angle ψ in terms of the time, which has already been obtained, we will denote as follows:

$$\psi = \theta_0(t+c_0) + K_1\sin\theta_0(t+c_0) + K_2\sin 2\theta_0(t+c_0) + \ldots.$$

Substituting this for ψ in the preceding formulæ, and putting in succession

$$\beta = \tfrac{1}{2}Ct, \qquad \beta = 90^\circ + \tfrac{1}{2}Ct,$$

we get

$$\sqrt{1-x}\,{\sin \atop \cos}\tilde{\omega} = \tfrac{1}{2}\big[\sqrt{1-\mathrm{b}} \mp \sqrt{1-\mathrm{a}}\big]{\sin \atop \cos}\big[(P_0+\tfrac{1}{2})\theta_0(t+c_0) + c$$

$$+ P_1\sin\theta_0(t+c_0) + P_2\sin 2\theta_0(t+c_0) + \ldots\big],$$

$$+ \tfrac{1}{2}\big[\sqrt{1-\mathrm{b}} \pm \sqrt{1-\mathrm{a}}\big]{\sin \atop \cos}\big[(P_0-\tfrac{1}{2})\theta_0(t+c_0) + c$$

$$+ Q_1\sin\theta_0(t+c_0) + Q_2\sin 2\theta_0(t+c_0) + \ldots\big],$$

$$\sqrt{1+x}\,{\sin \atop \cos}\tilde{\omega}' = \tfrac{1}{2}\big[\sqrt{1+\mathrm{b}} \mp \sqrt{1+\mathrm{a}}\big]{\sin \atop \cos}\big[(P_0+\tfrac{1}{2})\theta_0(t+c_0) + c'$$

$$+ P_1'\sin\theta_0(t+c_0) + P_2'\sin 2\theta_0(t+c_0) + \ldots\big],$$

$$+ \tfrac{1}{2}\sqrt{}\big[1+\mathrm{b} \pm \sqrt{1+\mathrm{a}}\big]{\sin \atop \cos}\big[(P_0-\tfrac{1}{2})\theta_0(t+c_0) + c'$$

$$+ Q_1'\sin\theta_0(t+c_0) + Q_2'\sin 2\theta_0(t+c_0) + \ldots\big].$$

Here we have put

$$
\begin{aligned}
P_0 &= N_0 + \frac{1}{2}\frac{C}{\theta_0}, \\
P_1 &= N_1 + (N_0 + \tfrac{1}{2})\, K_1, \\
P_2 &= \tfrac{1}{2}[N_2 + N_1 K_1 + 2(N_0 + \tfrac{1}{2})\, K_2], \\
Q_1 &= N_1 + (N_0 - \tfrac{1}{2})\, K_1, \\
Q_2 &= \tfrac{1}{2}[N_2 + N_1 K_1 + 2(N_0 - \tfrac{1}{2})\, K_2], \\
P_1' &= N_1' + (N_0 + \tfrac{1}{2})\, K_1, \\
P_2' &= \tfrac{1}{2}[N_2' + N_1' K_1 + 2(N_0 + \tfrac{1}{2})\, K_2], \\
Q_1' &= N_1' + (N_0 - \tfrac{1}{2})\, K_1, \\
Q_2' &= \tfrac{1}{2}[N_2' + N_1' K_1 + 2(N_0 - \tfrac{1}{2})\, K_2].
\end{aligned}
$$

It is evident from the equivalent of $\sin \nu \cos \gamma$ derived from these equations that $c' = c$ or $c' = c + 180°$, according as

$$H(b) = D_0 + D_1 b + D_2 b^2 + D_3 b^3 + \ldots = \pm \sqrt{1 - b^2}$$

is positive or negative. Hence the latter of the two equations may be written

$$
\begin{aligned}
\sqrt{1 + x}\,{\sin \atop \cos}\,\tilde{\omega}' = {} & \pm \tfrac{1}{2}[\sqrt{1 + \mathrm{b}} \mp \sqrt{1 + \mathrm{a}}]\,{\sin \atop \cos}\,[(P_0 + \tfrac{1}{2})\,\theta_0 (t + c_0) + c \\
& \qquad + P_1' \sin \theta_0 (t + c_0) + P_2' \sin 2\theta_0 (t + c_0) + \ldots] \\
& \pm \tfrac{1}{2}[\sqrt{1 + \mathrm{b}} \pm \sqrt{1 + \mathrm{a}}]\,{\sin \atop \cos}\,[(P_0 - \tfrac{1}{2})\,\theta_0 (t + c_0) + c \\
& \qquad + Q_1' \sin \theta_0 (t + c_0) + Q_2' \sin 2\theta_0 (t + c_0) + \ldots],
\end{aligned}
$$

where the upper or lower of the newly introduced ambiguous signs is taken according as $H(b)$ is positive or negative.

Let us put

$$
\begin{aligned}
\chi &= (P_0 + \tfrac{1}{2})\,\theta_0 (t + c_0) + c, \\
\chi' &= (P_0 - \tfrac{1}{2})\,\theta_0 (t + c_0) + c, \\
\triangle &= \tfrac{1}{2}[\sqrt{1 - \mathrm{b}} \mp \sqrt{1 - \mathrm{a}}], \\
\triangle_1 &= \tfrac{1}{2}[\sqrt{1 - \mathrm{b}} \pm \sqrt{1 - \mathrm{a}}], \\
\triangle' &= \pm \tfrac{1}{2}[\sqrt{1 + \mathrm{b}} \mp \sqrt{1 + \mathrm{a}}], \\
\triangle_1' &= \pm \tfrac{1}{2}[\sqrt{1 + \mathrm{b}} \pm \sqrt{1 + \mathrm{a}}].
\end{aligned}
$$

Then

$$
\begin{aligned}
\sqrt{1-x}\,\frac{\sin}{\cos}\,\tilde{\omega} = {} & [\triangle(1-\tfrac{1}{4}P_1^2)+\tfrac{1}{2}\triangle_1 Q_1]\frac{\sin}{\cos}\chi \\
& +[\triangle_1(1-\tfrac{1}{4}Q_1^2)-\tfrac{1}{2}\triangle P_1]\frac{\sin}{\cos}\chi' \\
& +[\tfrac{1}{2}\triangle P_1+\triangle_1(\tfrac{1}{8}Q_1^2+\tfrac{1}{2}Q_2)]\frac{\sin}{\cos}(2\chi-\chi') \\
& +[-\tfrac{1}{2}\triangle_1 Q_1+\triangle(\tfrac{1}{8}P_1^2-\tfrac{1}{2}P_2)]\frac{\sin}{\cos}(2\chi'-\chi) \\
& +\triangle(\tfrac{1}{8}P_1^2+\tfrac{1}{2}P_2)\frac{\sin}{\cos}(3\chi-2\chi') \\
& +\triangle_1(\tfrac{1}{8}Q_1^2-\tfrac{1}{2}Q_2)\frac{\sin}{\cos}(3\chi'-2\chi)\,,
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{1+x}\,\frac{\sin}{\cos}\,\tilde{\omega}' = {} & [\triangle'(1-\tfrac{1}{4}P_1'^2)+\tfrac{1}{2}\triangle_1' Q_1']\frac{\sin}{\cos}\chi \\
& +[\triangle_1'(1-\tfrac{1}{4}Q_1'^2)-\tfrac{1}{2}\triangle' P_1']\frac{\sin}{\cos}\chi' \\
& +[\tfrac{1}{2}\triangle' P_1'+\triangle_1'(\tfrac{1}{8}Q_1'^2+\tfrac{1}{2}Q_2')]\frac{\sin}{\cos}(2\chi-\chi') \\
& +[-\tfrac{1}{2}\triangle_1' Q_1'+\triangle'(\tfrac{1}{8}P_1'^2-\tfrac{1}{2}P_2')]\frac{\sin}{\cos}(2\chi'-\chi) \\
& +\triangle'(\tfrac{1}{8}P_1'^2+\tfrac{1}{2}P_2')\frac{\sin}{\cos}(3\chi-2\chi') \\
& +\triangle_1'(\tfrac{1}{8}Q_1'^2-\tfrac{1}{2}Q_2')\frac{\sin}{\cos}(3\chi'-2\chi)\,.
\end{aligned}
$$

It is evident that $e\frac{\sin}{\cos}\tilde{\omega}$ and $e'\frac{\sin}{\cos}\tilde{\omega}'$ can be expressed in series of the same form.

In applying to Jupiter and Saturn these equations, it is found that by varying the value of K,

$$K\frac{dD_0}{dK}=+0.0101629,\qquad K\frac{dD_1}{dK}=+0.0009178,\qquad K\frac{dD_2}{dK}=-0.0050568.$$

Also

$$\log\left[-\sqrt{(1-\mathrm{a})(1-\mathrm{b})}\right]=9.9574334n,\qquad \log\sqrt{(1+\mathrm{a})(1+\mathrm{b})}=9.4074864,$$

$$
\begin{aligned}
L_0 &= +0.1672972, & L_0' &= +0.1655301, \\
L_1 &= +0.0028107, & L_1' &= -0.0012760, \\
L_2 &= -0.0000071, & L_2' &= -0.0000250.
\end{aligned}
$$

Whence

$$N_0 = +\,0.1667632, \qquad N_0' = +\,0.1667632,$$
$$N_1 = -\,0.0021649, \qquad N_1' = +\,0.0009746,$$
$$N_2 = -\,0.0000021, \qquad N_2' = -\,0.0000074.$$

Also

$$P_0 = +\,0.6837293, \qquad P_1' = -\,786''.82,$$
$$P_1 = -\,1434''.41, \qquad P_2' = +\,2''.54,$$
$$P_2 = +\,5''.41, \qquad Q_1' = +\,694''.76,$$
$$Q_1 = +\,47''.17, \qquad Q_2' = -\,3''.50,$$
$$Q_2 = -\,0''.63.$$

$$\log \triangle = 9.5750158, \qquad \log \triangle' = 9.8634412n,$$
$$\log \triangle_1 = 0.0101623, \qquad \log \triangle_1' = 9.7217366,$$
$$(P_0 + \tfrac{1}{2})\,\theta_0 = 22''.55981, \qquad (P_0 - \tfrac{1}{2})\,\theta_0 = 3''.50156.$$

$$\begin{aligned}\sqrt{1-x}\genfrac{}{}{0pt}{}{\sin}{\cos}\tilde{\omega} = &+ 0.3759635\genfrac{}{}{0pt}{}{\sin}{\cos}\chi &&+ 1.0249824\genfrac{}{}{0pt}{}{\sin}{\cos}\chi' \\ &- 0.0013085\genfrac{}{}{0pt}{}{\sin}{\cos}(2\chi-\chi') &&- 0.0001196\genfrac{}{}{0pt}{}{\sin}{\cos}(2\chi'-\chi) \\ &+ 0.0000072\genfrac{}{}{0pt}{}{\sin}{\cos}(3\chi-2\chi') &&+ 0.0000016\genfrac{}{}{0pt}{}{\sin}{\cos}(3\chi'-2\chi),\end{aligned}$$

$$\begin{aligned}\sqrt{1+x}\genfrac{}{}{0pt}{}{\sin}{\cos}\tilde{\omega}' = &- 0.7293089\genfrac{}{}{0pt}{}{\sin}{\cos}\chi &&+ 0.5255160\genfrac{}{}{0pt}{}{\sin}{\cos}\chi' \\ &+ 0.0013889\genfrac{}{}{0pt}{}{\sin}{\cos}(2\chi-\chi') &&- 0.0008842\genfrac{}{}{0pt}{}{\sin}{\cos}(2\chi'-\chi) \\ &- 0.0000058\genfrac{}{}{0pt}{}{\sin}{\cos}(3\chi-2\chi') &&+ 0.0000052\genfrac{}{}{0pt}{}{\sin}{\cos}(3\chi'-2\chi).\end{aligned}$$

The value of c is found to be

$$c = 340^\circ\ 8'\ 50''.26.$$

Hence the expressions for the two arguments are

$$\chi = 308^\circ\ 13'\ 15''.13 + 22''.55981t,$$
$$\chi' = 31^\circ\ 4'\ 5''.98 + 3''.50156t.$$

The following expressions for e and e' were obtained :

$$\frac{e}{\sqrt{1-x}} = [8.6282138] \sqrt{(1 - [6.5410419] \cos \psi)},$$

$$\frac{e'}{\sqrt{1+x}} = [8.8231642] \sqrt{(1 + [6.9312571] \cos \psi)},$$

$$\frac{e}{\sqrt{1-x}} = [8.6282135] \{1 - [6.24001] \cos (\chi - \chi') + [3.7900] \cos 2 (\chi - \chi')\},$$

$$\frac{e'}{\sqrt{1+x}} = [8.8231648] \{1 + [6.63023] \cos (\chi - \chi') - [4.1982] \cos 2 (\chi - \chi')\}.$$

By means of these we can pass to the expressions for the following functions :

$$\begin{aligned} e \frac{\sin}{\cos} \varpi = &\; 0.01596822 \frac{\sin}{\cos} \chi &&+ 0.04354278 \frac{\sin}{\cos} \chi' \\ &- 0.00005696 \frac{\sin}{\cos} (2\chi - \chi') &&- 0.00000886 \frac{\sin}{\cos} (2\chi' - \chi) \\ &+ 0.00000031 \frac{\sin}{\cos} (3\chi - 2\chi') &&+ 0.00000009 \frac{\sin}{\cos} (3\chi' - 2\chi), \end{aligned}$$

$$\begin{aligned} e' \frac{\sin}{\cos} \varpi' = &- 0.04852990 \frac{\sin}{\cos} \chi &&+ 0.03496407 \frac{\sin}{\cos} \chi' \\ &+ 0.00008205 \frac{\sin}{\cos} (2\chi - \chi') &&- 0.00005134 \frac{\sin}{\cos} (2\chi' - \chi) \\ &- 0.00000033 \frac{\sin}{\cos} (3\chi - 2\chi') &&+ 0.00000031 \frac{\sin}{\cos} (3\chi' - 2\chi). \end{aligned}$$

It will be observed that these expressions are as convergent as could be wished. The form of these integrals being discovered, another and more direct method of arriving at them is suggested. The coefficients being assumed as indeterminate as well as the rates of movement of the two arguments together with the constants which complete the values of the latter, the expressions could be substituted in the differential equations, and thus would arise twelve equations of condition, which along with the value of the four variables at the origin of time would determine the sixteen unknowns involved. But on trial it seems this way of proceeding would necessitate as long computations as the method we have followed.

In conclusion, it may be observed that, if terms arising from the squares and higher powers of the masses were taken into consideration, the form of this investigation would not thereby be changed; the only effect produced would be that the values of the various constants involved would receive slight modifications.

SOLUTIONS OF EXERCISES.

138

About the vertices of an equilateral triangle three spheres are drawn with radii equal to the side of the triangle. Find the volume common to them all.

[*W. M. Thornton.*]

SOLUTION.

Let ABC be the triangle of the centres, O its ortho-centre, and N one of the apices of the solid. Draw BO to meet the surface in Q. The planes NOA, NOQ, AOQ, and the sphere-surface ANQ bound one-twelfth of the solid. The area of the sphere-surface ANQ is the difference between the area of the zone-surface ACN, whose angle ACN is arc tan $2\sqrt{2}$, and that of the right spherical triangle QCN, whose angles are arc tan $2\sqrt{2}$ and arc cot $2\sqrt{2}$.

The excess of this triangle is

$$E = \text{arc tan } 2\sqrt{2} + \text{arc cot } \sqrt{2} - \tfrac{1}{2}\pi.$$

Its area is

$$\Delta = ER^2,$$

R being the radius of the sphere. The area of the zone-piece is

$$Z = \tfrac{1}{2} R^2 \text{ arc tan } 2\sqrt{2}.$$

Hence the curved surface of the portion of the solid under consideration is

$$\begin{aligned} Z - \Delta &= R^2 (\tfrac{1}{2}\pi - \text{arc cot } \sqrt{2} - \tfrac{1}{2} \text{ arc tan } 2\sqrt{2}) \\ &= R^2 (\text{arc tan } \sqrt{2} - \tfrac{1}{2} \text{ arc tan } 2\sqrt{2}). \end{aligned}$$

The volume of the spherical pyramid which has this surface for its base is

$$\tfrac{1}{3} R^3 (\text{arc tan } \sqrt{2} - \tfrac{1}{2} \text{ arc tan } 2\sqrt{2}).$$

The volume of the cone-segment whose base is NOA and apex B is

$$\tfrac{1}{3} R^3 (\tfrac{1}{4} \text{ arc tan } 2\sqrt{2} - \tfrac{1}{24}\sqrt{2}).$$

The difference between these volumes is one-twelfth the volume of the solid common to three spheres. Hence

$$V = R^3 (\tfrac{1}{12}\sqrt{2} + 4 \text{ arc tan } \sqrt{2} - 3 \text{ arc tan } 2\sqrt{2}).$$

[*W. H. Echols.*]

271

If

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & \dots & h_1 \\ a_2 & b_2 & c_2 & \dots & h_2 \\ a_3 & b_3 & c_3 & \dots & h_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & h_n \end{vmatrix}, \text{ and } D = \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_2 & h_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_3 & h_3 \end{vmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_n & c_n \end{vmatrix} & \dots & \begin{vmatrix} a_1 & h_1 \\ a_n & h_n \end{vmatrix} \end{vmatrix};$$

then will $$D = a_1^{n-2}\Delta.$$ [*T. M. Blakslee.*]

SOLUTION.

This exercise is given on p. 77 of Muir's Theory of Determinants. It may be got from the result of section 53 of that work; or it may be derived by a process analogous to that of the section referred to, as follows: Multiply each column after the first by a_1; add to each element of the second column, thus multiplied, $-b_1$ times the corresponding element of the first column; to each element of the new third column $-c_1$ times the corresponding element of the first column; ... to each element of the new nth column $-h_1$ times the corresponding element of the first column; and we have

$$a_1^{n-1}\Delta = \begin{vmatrix} a_1 & 0 & 0 & \dots & 0 \\ a_2 & -a_2b_1 + a_1b_2 & -a_2c_1 + a_1c_2 & \dots & -a_2h_1 + a_1h_2 \\ a_3 & -a_3b_1 + a_1b_3 & -a_3c_1 + a_1c_3 & \dots & -a_3h_1 + a_1h_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & -a_nb_1 + a_1b_n & -a_nc_1 + a_1c_n & \dots & -a_nh_1 + a_1h_n \end{vmatrix}$$

$$= a_1 \begin{vmatrix} |a_1\ b_2| & |a_1\ c_2| & \dots & |a_1\ h_2| \\ |a_1\ b_3| & |a_1\ c_3| & \dots & |a_1\ h_3| \\ \dots & \dots & \dots & \dots \\ |a_1\ b_n| & |a_1\ c_n| & \dots & |a_1\ h_n| \end{vmatrix} = a_1 D;$$

$$\therefore a_1^{n-2}\Delta = D.$$

[*W. B. Richards.*]

Also solved by L. G. Weld and the proposer.

276

Show that the attraction of a finite mass on one of its points is finite.

[*A. Hall.*]

SOLUTION.

Take the origin at the point attracted, and let $k\,dx\,dy\,dz$ be the element of mass, V the potential for a particle of the body, and r the distance of the particle from the origin. We have

$$V = \int\int\int \frac{k\,dx\,dy\,dz}{r}.$$

Put

$$x = r\cos u, \qquad y = r\sin u\cos\lambda, \qquad z = r\sin u\sin\lambda;$$

then

$$dx\,dy\,dz = r^2\sin u\,du\,d\lambda\,dr,$$

and

$$V = \int\int\int kr\sin u\,du\,d\lambda\,dr.$$

The limits of integration are $u = 0$, to $u = \pi$; $\lambda = 0$, to $\lambda = 2\pi$; and $r = 0$, to the limits of the body. For the component of the force in the axis of x we have

$$X = \frac{\partial V}{\partial x} = \int\int\int \frac{kx\,dx\,dy\,dz}{r^3} = \int\int\int k\cos u\sin u\,du\,d\lambda\,dr,$$

with similar values for Y and Z. These components are finite, and therefore the resultant is finite. See Gauss, *Allgemeine Lehrsätze, etc.*

[*A. Hall.*]

EXERCISES.

306

A body at distance r from the sun is moving with velocity v. Prove that the major axis of the orbit described is parallel to the direction of motion if, and only if, the velocity is "circular velocity for the distance r."

[*Ellery W. Davis.*]

307

If a horizontal beam of length $2a$ is supported at each end, and has a load in the form of an isosceles triangle, base $2a$, height b, a unit's thickness throughout, and heaviness unity; show that the deflection of the beam due to this triangular load is $\dfrac{2a^4b}{15EI}$.

[*T. U. Taylor.*]

CONTENTS.

ANNALS OF MATHEMATICS.

Terms of subscription: \$2 for a volume of at least 200 pages: THE MONEY MUST IN ALL CASES ACCOMPANY THE ORDER. All drafts and money orders should be made payable to the order of ORMOND STONE, University Station, Charlottesville, Va., U. S. A.

PUBLICATIONS RECEIVED.

Die Bahn des Periodischen Kometen Winnecke in den Jahren 1858–1886. II Thiel. Von Dr. Edward Freiherrn von Haerdtl.

The Orbit of Swift's Comet 1880 V, Determined by Gibbs's Vector Method. By W. Beebe and A. W. Phillips. (Astronomical Journal.)

Bidrag till de matematiska studiernas historia i Sverige under femtonhundratalet. Af Gustaf Eneström. (Vetenskaps-Akademiens Forhandlingar.)

On the Reduction of Pendulum Observations. By Erasmus Darwin Preston. (Philosophical Society of Washington.)

Assumption and Fact in the Theories of Solar and Stellar Proper Motions. By John Robie Eastman. (Philosophical Society of Washington.)

The Mecanique Celeste of Laplace, and its Translation with a Commentary by Bowditch. By Joseph Lovering. (Proceedings of the American Academy.)

The Mathematical Theories of the Earth. By Robert Simpson Woodward. (Proceedings of the American Association for the Advancement of Science.)

Programme d'un Cours Universitaire d'Historie das Mathematiques par G. Eneström.

Logarithmic Tables. By George William Jones.

Publications of Carleton College Observatory. 1. Catalogue of 644 Comparison Stars Observed with the Repsold Meridian Circle during the Years 1887 to 1889, and Prepared for Publication under the Direction of William W. Payne, by Herbert C. Wilson.

Publications of the Cincinnati Observatory. Micrometrical Measurements of Double Stars, 1882–6, made with the 11-inch Equatorial. Prepared for Publication by H. C. Wilson, Assistant Astronomer.

Considerazioni sul moto rotatorio del pianeta Venere. Noto del M. E. G. V. Schiaparelli.

On the Computation of the Equation of the Centre in Elliptical Orbits of Moderate Eccentricities. A simple Solution of Kepler's Problem. By A. Marth. (Monthly Notices of the Royal Astronomical Society.)

My Class in Geometry. By George Iles. (Popular Science Monthly.)

Two Auxiliary Tables for the Solution of Kepler's Problem. Ephemerides of the Satellites of Saturn, 1890–91. Ephemeris of the Satellite of Neptune, 1890–91. By A. Marth. (Monthly Notices of the Royal Astronomical Society.)

GIBSON BROS., PRINTERS AND BOOKBINDERS, WASHINGTON, D. C.